Binary Structures

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Abstract

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Binary operations

The essence of an “operation" is that two things are combined to form a third thing of the same kind. For example, composition of functions combine two functions to give another function. Ordinary multiplication, addition, and subtraction combine two numbers to give another number.

Definition 1. A **binary operation** on a set G is a function

$$*\colon G\times G\mapsto G.$$

Example 1. Addition is an operation on the set $\mathbb{Z}_+$ whose value at a pair (x,y) is $x+y$.

Example 2. Subtraction is an operation on the set $\mathbb{Z}$ whose value at a pair (x,y) is $x-y$.

Example 3. Multiplication is an operation on the set $\mathbb{Z}_+$ whose value at a pair (x,y) is xy.

Example 4. let $\mathscr{M}$ be the set of all real valued functions of a real variable, that is, all functions $f\colon\mathbb{R}\mapsto\mathbb{R}$. Then composition is an operation on $\mathscr{M}$ whose value at a pair (f,g) is $f\circ g$.

In more detail, a binary operation assigns an element $*(x,y)$ in G to each ordered pair (x,y) of elements in G. It is more natural to write $x*y$ instead of $*(x,y)$; thus, composition of functions is the function $(g,f)\mapsto f\circ g$; multiplication, addition, and subtraction are, respectively, the functions $(x,y)\mapsto xy$, $(x,y)\mapsto x+y$, and $(x,y)\mapsto x-y$. The examples of composition and subtraction show why we want ordered pairs, for $x*y$ and $y*x$ may be distinct.

Definition 2. Suppose that $*$ is a binary operation on a set G and H is a subset of G. If the restriction of $*$ to H is a binary operation on H, i.e., for all $x,\, y \in H$ $x * y \in H$, then H is said to be **closed** under $*$. The binary operation on H given by restricting $*$ to H is the **induced operation** of $*$ on H.

Commutative and associative operations

Definition 3. A binary operation $*$ on a set G is **commutative** if for all $x,\, y \in G$, $x * y = y * x$.

Example 5. Addition and multiplication on $\mathbb{Z}_+$ are commutative, while subtraction on $\mathbb{Z}$ and composition of functions on $\mathscr{M}$ are non-commutative.

Definition 4. A binary operation $*$ on a set G is **associative** if for all $x,\, y,\, z \in G$ we have $x * (y * z) = (x * y) * z$.

Example 6. Addition and multiplication on $\mathbb{Z}_+$ are associative. Subtraction on $\mathbb{Z}$ is not associative. Composition of functions on $\mathscr{M}$ is associative.

Binary structures

Definition 5. A **binary algebraic structure** is a set S together with a binary operation $*$ on S.

Definition 6. Let $(S, *)$ and $(S', *')$ be binary algebraic structures. An **isomorphism** of $(S, *)$ with $(S', *')$ is a one-to-one correspondence $\varphi \colon S \mapsto S'$ such that for all $x,\; y \in S$ the following **homomorphism property** holds:

$$\varphi(x * y) = \varphi(x) *' \varphi(y).$$

If such a map φ exists, we say that $(S, *)$ and $(S', *')$ (or just S and S') are **isomorphic binary structures**, and denote this by $S \simeq S'$.

Example 7. Let U be the unit circle in the complex plane. The map $\varphi(\theta) = e^{i\theta}$ maps $\mathbb{R}_{2\pi}$ onto U and establishes an isomorphism between binary structures $(\mathbb{R}_{2\pi}, +_{2\pi})$ and $(U, \cdot)$.

Guidelines for proving the isomorphism of binary structures

There exist only one way to prove that two binary structures S and S' are isomorphic: propose a map φ and prove that your map satisfies Definition 6. So, you need to do **four** steps.

① Write a formula for φ.

② Prove that φ is one-to-one (injective).

③ Prove that $\varphi[S] = S'$ (φ is surjective).

④ Prove the homomorphism property.

Example 8. Let $n \in \mathbb{Z}^+$, and let $(S, *) = (\mathbb{Z}_n, +_n)$ and $(S', *') = (U_n, \cdot)$. Define $\varphi \colon \mathbb{Z}_n \mapsto U_n$ as

$$\varphi(k) = e^{2k\pi i/n}.$$

If $\varphi(k) = \varphi(l)$, then $e^{2k\pi i/n} = e^{2l\pi i/n}$. Taking the natural logarithm, we have $2k\pi i/n = 2l\pi i/n$, so $k = l$.

If $\theta \in U_n$, then $\theta = 2k\pi/n$ for $k \in \mathbb{Z}_n$, and $\varphi(k) = e^{2k\pi i/n} \in U_n$. Thus φ is onto U_n.

For k, $l \in \mathbb{Z}_n$ we have $\varphi(k +_n l) = e^{2(k+_n l)\pi i/n} = e^{2k\pi i/n} \cdot e^{2l\pi i/n} = \varphi(k) \cdot \varphi(l)$.

Guidelines for proving the non-isomorphism of binary structures

There exist only one way to prove that two binary structures S and S' are not isomorphic: find a property that must be shared by any isomorphic structures (**structural property**) but distinguishes S and S'.

Example 9. Because any isomorphism φ is a one-to-one correspondence, two isomorphic binary structures must have the same cardinality. Thus, the binary structures $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic.

Example 10. We have $|\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$. However, an equation $2 \cdot x = 1$ has a solution $x = 1/2$ in $\mathbb{Q}$, but no solutions in $\mathbb{Z}$. Therefore the binary structures $(\mathbb{Q}, \cdot)$ and $(\mathbb{Z}, \cdot)$ are not isomorphic.