Exercise assignment 2

Each exercise assignment has two parts. The first part consists of 5 elementary problems for a maximum of 10 points from each assignment. For the second part consisting of problems in applications you are to choose *one* of two problems to solve. This part can give up to 5 points from each assignment. The first part consists of elementary questions to make sure that you have understood the basic material of the course while the second part consists of larger application examples.

Solutions can either be submitted by mail to *christopher.engstrom@mdh.se* or *jonas.osterberg@mdh.se* or alternatively you can submit handwritten solutions in the envelope outside of room $U3 - 185$ before 23.59 on Sunday 17st of May.

Each exercise assignment can give a maximum of 15 points, to pass you will need at least 20 points total from both the assignments. If you do not get enough points from the assignments you will be given the opportunity to complement your solutions to reach a passing grade at a later time.

Part 1

In the first part you are to solve and hand in solutions to the questions. You are allowed to use computer software to check your results, but your calculations as well as your result should be included in the answers for full points.

1.1

Consider the three column vectors

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = (10,-2,-2), \quad \mathbf{v}_3 = (13,4,-8).$$

a) (1) Find an orthogonal basis $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ for $\mathbb{R}^3$ with $\mathbf{b}_1 = \mathbf{v}_1$ by using the Gram–Schmidt process on $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Show your steps.

b) (1) Using the results of part (a), let Q be the matrix with column vectors $\mathbf{b}_j$ and A be the matrix with column vectors $\mathbf{v}_j$:

$$\mathsf{Q} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix},$$
$$\mathsf{A} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}.$$

Find a (right triangular) matrix R such that

$$\mathsf{A} = \mathsf{QR}.$$

x	1	3	5	6
y	1	2	4	6

Table 1: xy-points of some data.

1.2

Consider the data in the table below:

a) (1) Use the least squares method to approximate a line $y = ax + b$ to the data given in table 1 in least square sense.

b) (1) Try to fit a polynomial of second degree to the data in table 1. What result do you get?

1.3

a) (1) Give a definition of the *Kroenecker product*, $\otimes$, there are two alternatives.

b) (1) Calculate

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

1.4

a) (1) Give a definition of a *linear transformation.*

b) (1) Let T be a transformation acting on functions such that for any Riemann integrable function $f : \mathbb{R} \to \mathbb{R}$ the function Tf is defined by

$$(Tf)(x) = \int_0^x f(t)\ dt.$$

Use the properties of (Riemann) integrals and the definition of a linear transformation and show that the transformation T is in fact linear.

1.5 For those who took the course HT14

Note that there are *two* versions of this problem 1.5, and you are to choose *one* of them to solve.

Consider 6 observations out of two classes with 2 measured variables each:

$$\begin{bmatrix} 4 & 2 & 5 & 1 & 3 & 4 \\ 1 & 2 & 3 & 3 & 3 & 4 \\ c1 & c1 & c1 & c2 & c2 & c2 \end{bmatrix}$$

where each row represent one variable (last denoting their class belonging) and each column represent one observation. Our aim is to use Linear discriminant analysis in order to be able to classify new observations for which we do no know their class already.

a) (1) For each class: calculate corresponding covariance matrix S_1, S_2 (within class scatter).

b) (1) Use LDA to find the line w^* which when projected upon best separates the two classes. Plot the 6 points and the line w^* (by hand or using a computer does not matter).

1.5 For those who took the course HT13 or earlier

Note that there are *two* versions of this problem 1.5, and you are to choose *one* of them to solve.

- (1) The Fibonacci sequence is usually defined by the recursion

$$\begin{cases} f_{n+1} = f_n + f_{n-1} & n \geq 1 \\ f_0 = 0 \\ f_1 = 1 \end{cases}$$

  We can also define the recursion on the form

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = M \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix},$$

  where M is a 2×2 matrix, and $f_0 = 0$ and $f_1 = 1$ as usual. Find the matrix M.

- (1) Repeating the matrix recursion in the first part of this problem we get the matrix function

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = M^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad n \geq 0.$$

  Use diagonalization of the matrix M to find a formula for the n:th Fibonacci number.

Part 2

In this second part you are to choose **ONE** example where you attempt to solve the questions presented. If you hand in answers to more than one choice you will get points corresponding to the choice which would give the least total points.

You are allowed to use computer software (and depending on which option you choose might be needed for some of the questions). If you are using a computer for some calculations you should set up the problem and present how it could theoretically be solved by hand. For example you could write 'I solved the linear system $\mathsf{A}\mathbf{x} = \mathbf{b}$ using Matlabs `fsolve`, another method would have been to use Gaussian elimination and solving the resulting triangular system'.

Using LDA to predict bankruptcy

This is a simplified version of the Altman Z-scores used to predict bankruptcy using fewer variables but the method itself remains the same. The aim of the model is to predict if a company will go bankrupt within two years given some data of the company. We will use the following three variables:

- T1: Retained Earnings / Total Assets. A measure of how profitable the company is.
- T2: Earnings Before Interest and Taxes / Total Assets. A measure of operating efficiency
- T3: Sales/Total Assets.

We have data of 8 companies total, 4 of which went bankrupt within two years given by the following table:

T_1	0.24	0.10	−0.04	0.30	−0.05	−0.20	0.05	−0.15
T_2	0.07	−0.01	−0.06	0.05	0.02	−0.15	−0.2	−0.35
T_3	1.58	1.62	1.84	1.4	0.9	1.2	1.6	1.2
	C_1	C_1	C_1	C_1	C_2	C_2	C_2	C_2

Here C_1 is the class of companies that did not default and C_2 is the class of companies that did default within 2 years.

a) (1) For each class: calculate corresponding covariance matrix S_1, S_2 (within class scatter).

b) (1) Use LDA to find the line w^* which when projected upon best separates the two classes. Find a suitable point on this line which can be used to discriminate between the two classes.

c) (1) Interpret your results, what is the most important variable for a company that wants to avoid bankruptcy?

Now that we have our model we are interested in predicting if any of two new companies are likely to go bankrupt within two years.

T_1	0.05	−0.02
T_2	0.03	0.04
T_3	1.6	1.1

d) (1) Classify these two new companies. Are any of them likely to go bankrupt?

e) (1) Classify all original companies and calculate the confusion matrix. Interpret the results.

Using Least Squares Method to analyze a standing wave in a half open pipe

A half open pipe as that depicted in Figure 1 can contain a standing wave in the pressure of the air filling the tube. The wavelengths of standing waves in this tube can be only (ideally) from a discrete set of wavelengths depending only on the length of the tube. Standing waves of more that one wavelength can be present at the same time in the tube. The wave of the shortest wavelength is called the *Fundamental mode* or *First Harmonic* and the subsequent waves are called second *Harmonic*, third, and so on, in order of increasing wavelength. The pressure in the open end of the tube will be the outside pressure (room pressure).

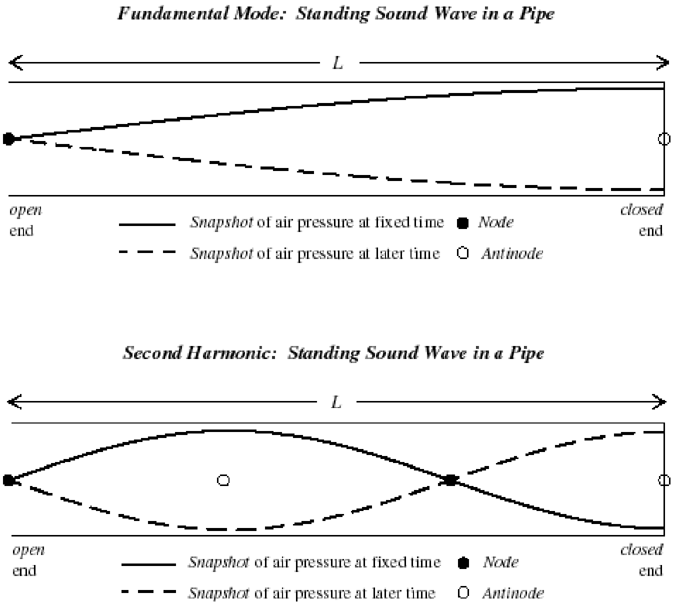

Figure 1: The first two harmonics of standing waves in a half-open pipe. The open end always correspond to a node (no change in pressure) and the closed end is an antinode (the most change in pressure over time). Figures borrowed from `http://hep.physics.indiana.edu/~rickv/Standing_Sound_Waves.html`.

An experiment was conducted in which at one point in time, the air pressure was measured at 9 different points inside a one-meter half-open tube, using 9 different pressure sensors. The measured pressures in the pipe is shown in Table 2.

x (m)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
p (bar)	1.024	1.045	1.075	1.063	1.057	1.045	1.029	1.015	1.004

Table 2: Measured points (x, p) where x is the distance in meters from the open end of the pipe and p is the air pressure in bars.

a) (1) Assume that the pressures from Table 2 are solely due to standing waves. Use the basis functions $f_1(x) = 1$ and $f_2(x) = \sin\left(\frac{\pi}{2}x\right)$ and make a least squares fit of the pressures in the tube from Table 2.

b) (1) We have no measurement of the air pressure *outside* of the tube. From the fit in part (a), what does this air pressure seem to be? What is the amplitude of the first harmonic.

c) (1) We have good reasons to believe that it is not only the fundamental node present in the tube. Lets use the second harmonic as well. That is, use the least squares method with the basis functions $f_1(x) = 1$, $f_2(x) = \sin\left(\frac{\pi}{2}x\right)$ and $f_3(x) = \sin\left(\frac{3\pi}{2}x\right)$.

d) (1) Discuss the results from part (c). Is this fit better/worse?

e) (1) At a previous time an experiment went wrong and the tube was overloaded with a very high amplitude in the second harmonic. This is believed to have made the pressure sensors a bit unreliable. Use a weighted least squares fit of the first two harmonics, giving a lower weight for the sensors that you believe were the most damaged by the overload.