Exercise assignment 1

Each exercise assignment has two parts. The first part consists of $3-5$ elementary problems for a maximum of 10 points from each assignment. For the second part consisting of problems in applications you are to choose *one* of three problems to solve. This part can give up to 5 points from each assignment. The first part consists of elementary questions to make sure that you have understood the basic material of the course while the second part consists of larger application examples.

Solutions can either be submitted by mail to *christopher.engstrom@mdh.se* or *jonas.osterberg@mdh.se* or alternatively you can submit handwritten solutions in the envelope outside of room $U3-185$ before 23.59 on Sunday 17th of May.

Each exercise assignment can give a maximum of 15 points, to pass you will need at least 20 points total from both the assignments. If you do not get enough points from the assignments you will be given the opportunity to complement your solutions to reach a passing grade.

1 Part 1

In the first part you are to solve and hand in solutions to the questions. You are allowed to use computer software to check your results, but your calculations as well as your result should be included in the answers for full points.

1.1

a) (1) Use the characteristic polynomial $\det(\mathsf{I}\lambda - \mathsf{A})$ to find the eigenvalues of A.

b) (1) Calculate the determinant of A

$$\mathsf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

1.2

a) (1) Define the adjacency matrix for a directed graph.

b) (1) Given the adjacency matrix A below, draw corresponding directed graph.

$$\mathsf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

1.3

Consider the stochastic matrix M below.

$$\mathsf{M} = \begin{bmatrix} 0 & 0.4 & 0.6 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0.6 & 04 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

a) (1) Write out the graph describing the transitions of this Markov chain.

b) (1) If we start in the first state, calculate the probability that we are in state 3 after 4 steps in the Markov chain described by M.

1.4

Consider matrix M in the previous problem.

a) (1) Give a definition of irreducible and primitive matrices.

b) (1) Prove that M is or isn't irreducible and primitive.

c) (1) Without calculating the eigenvalues: Use the Perron-Frobenius theorem to estimate the value of the spectral radius of M and show if there could be more than one eigenvalue on the spectral radius.

1.5

We consider the matrices $\mathsf{P}, \mathsf{A}, \mathsf{L}, \mathsf{U}$ with LUP-factorization $\mathsf{PA} = \mathsf{LU}$.

$$\mathsf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; \mathsf{A} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -2 & 0 & 4 \end{bmatrix}, \; \mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 2 \end{bmatrix}, \; \mathsf{U} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

a) (1) Use the LUP-factorization above to solve $\mathsf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [-2 \; 1 \; -3]^\top$.

Part 2

In this second part you are to choose **ONE** example where you attempt to solve the questions presented. If you hand in answers to more than one choice you will get points corresponding to the choice which would give the least total points.

You are allowed to use computer software (and depending on which option you choose might be needed for some of the questions). If you are using a computer for some calculations you should set up the problem and present how it could theoretically be solved by hand. For example you could write "I solved the linear system $Ax = b$ using Matlabs "fsolve", another method would have been to use Gaussian elimination and solving the resulting triangular system".

Credit rating of bonds

We consider how the credit rating of bonds change from one rating to another or defaulting. The bonds are rated once every year at the same time. From historical data we have found that the credit rating of a bond can be modelled using a markov chain with transition matrix P. We consider the 3 ratings A, B, C and default D.

	A	B	C	D
A	19	1	0	0
B	2	25	2	1
C	0	5	12	3

- (a) Create a stochastic matrix for this Markov chain model with transition probabilities $T_{i,j} = r_{ij}/r_i$ where $r_{i,j}$ is the number of observed transitions from credit rating i to credit rating j and r_i is the total number of observed transitions from state i. Default (D) should be represented by a single absorbing state.
- (b) What is the probability that a B rated bond will default within 3 years? $(P(X_3 = D|X_0 = B))$.
- (c) How many years on average will it take for a A rated Bond before it defaults?

The model above have one problem when we want to look at all the bonds, every bond is bound to eventually default and since we never introduce new bonds, eventually all bonds will have defaulted. But we want to say something about the long term as well. To do this we modify our model by replacing every defaulting bond with a new bond starting with a B rating the next year. This can be seen as a transition from D to B.

- (d) Change your stochastic matrix such that defaulted bonds are replaced as described above. Show that this new matrix is primitive.
- (e) Calculate the stationary distribution of this new Markov chain and interpret your results.

PCA in energy

The consumption of electrical energy for a household varies over time. It may depend on for example the time of day, time of week, time of month and time of the year. Different measures of this energy consumption can be related to each other. For example, the mean energy consumption in January and April may give clues for the mean consumption in September.

The energy consumption for ten households where measured in January, April and September, resulting in the matrix A:

$$A = \begin{bmatrix} 20 & 10 & 2 \\ 7 & 1 & 2 \\ 13 & 1 & 9 \\ 10 & 0 & 5 \\ 23 & 6 & 6 \\ 21 & 2 & 7 \\ 18 & 9 & 0 \\ 20 & 5 & 5 \\ 17 & 8 & 3 \\ 18 & 0 & 9 \end{bmatrix}$$

Each row in the matrix A is for one household and on the form (c_j, c_a, c_s) where c_j is the mean consumption in January, c_a is the mean consumption in April and c_s is the mean consumption in September, all measured in units of 100 Watts. We are going to use Principal Component Analysis to try to reduce this data from three dimensional to two dimensional. This will allow us to predict the consumption in one month if we know the consumption in the other two months.

- (a) Construct a new matrix B by centering the data in matrix A by subtracting the mean values of c_j, c_a and c_s from the respective columns in A.
- (b) Construct the 3×3 covariance matrix C for the matrix B.
- (c) Calculate the eigenvalues and the corresponding eigenvectors of the covariance matrix C.
- (d) Select two suitable eigenvectors from the three calculated in (c) and project the data on the subspace defined by this eigenvectors. Motivate you choice of eigenvectors. The resulting matrix D should be of the same dimensions as A and B. Now add the suitable means to each column in D to get the matrix E in the original coordinate system as that for the original matrix A.
- (e) Use an equation for the plane in which all points in E lie, expressed in the variables c_j, c_a and c_s to predict the energy consumption in September, when the energy consumption in January is 10 and the consumption in April is 3.