1

We see that both the numerator and the denominator has the limit 0 so we try L'Hospital's rule. Set $f(x) = \sin(x^2)$ and $g(x) = 1 - \cos(x)$. Then $f'(x) = 2x\cos(x^2)$ and $g'(x) = \sin(x)$. We see that we still have an ideterminate form of the type $\frac{0}{0}$ so we try L'Hospital again. We find that $f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$ and $g''(x) = \cos(x)$. Now we can compute the limit. Taking it all together we get:

$$\lim_{x\to 0} \frac{\sin(x^2)}{1-\cos(x)} = \lim_{x\to 0} \frac{2x\cos(x^2)}{\sin(x)} =$$
$$\lim_{x\to 0} \frac{2\cos(x^2) - 4x^2\sin(x^2)}{\cos(x)} = \frac{2}{1} = 2.$$

Answer: The limit is 2.

2

We take the derivative of both sides with respect to x. We get

$$y' + 2yy' = e^x \Leftrightarrow (1+2y)y' = e^x \Leftrightarrow y' = \frac{e^x}{1+2y}.$$

We use the assumption that $e^x = y + y^2$ to rewrite the expression for y':

$$y' = \frac{y+y^2}{1+2y}.$$

Answer: $\frac{dy}{dx} = \frac{y+y^2}{1+2y}$.

3

We use the change of variables $u = \sqrt{x+1}$. We see that $x = u^2 - 1$ and that $dx = 2udu$, using the standard way of writing change of variables.

$$\int_0^2 \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \int_1^{\sqrt{3}} \frac{u^2-1}{u} 2u\mathrm{d}u = 2\int_1^{\sqrt{3}} (u^2-1)\,\mathrm{d}u =$$
$$2\left[\frac{u^3}{3} - u\right]_1^{\sqrt{3}} = 2\left(\frac{3\sqrt{3}}{3} - \sqrt{3}\right) - 2\left(\frac{1}{3} - 1\right) = \frac{4}{3}.$$

Answer: The integral is equal to $\frac{4}{3}$.

4

We rewrite the equation:

$$y' = y^2 \cdot \sin(x) \Leftrightarrow \frac{y'}{y^2} = \sin(x) \Leftrightarrow -\frac{1}{y} = -\cos(x) + C \Leftrightarrow$$
$$y = \frac{1}{\cos(x) - C}.$$

Since $y(0) = \frac{1}{2}$ we must have that

$$\frac{1}{2} = \frac{1}{\cos(0) - C} = \frac{1}{1 - C} \Leftrightarrow C = -1.$$

Answer: The solution is $y(x) = \frac{1}{\cos(x)+1}$.

5

We use the root test. Set $a_n = (-1)^n e^{n-\sqrt{n}} x^n$. Then

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} e^{1-\frac{\sqrt{n}}{n}} |x| =$$
$$\lim_{n\to\infty} e^{1-\frac{1}{\sqrt{n}}} |x| = e^{1-0}|x| = e|x|.$$

By the ratio test the series converges if $e|x| < 1$ and diverges if $e|x| > 1$. The condition $e|x| < 1$ is equivalent to $|x| < \frac{1}{e}$.

Answer: The radius of convergence is $\frac{1}{e}$.