1

We note that direct substitution of $x = 0$ in the function gives $\frac{0}{0}$. We use l'Hospital's rule instead.

The derivative of the numerator is $2x - 1$ and the derivative of the denominator is e^x.

We see that

$$\lim_{x\to 0} \frac{2x-1}{e^x} = \frac{-1}{1} = -1.$$

By l'Hospital's theorem it is also true that

$$\lim_{x\to 0} \frac{x^2-x}{e^x-1} = -1.$$

Answer: The limit is -1.

2

We find that

$$f'(x) = e^x(x^2 + \sin(x)) + e^x(2x + \cos(x)) = e^x(x^2 + 2x + \cos(x) + \sin(x)).$$

We take the derivative of this to find $f''(x)$.

$$f''(x) = e^x(x^2 + 2x + \cos(x) + \sin(x)) + e^x(2x + 2 - \sin(x) + \cos(x)) =$$
$$= e^x(x^2 + 4x + 2 + 2\cos(x)).$$

Answer: See above.

3

We make the change of variable $u = \ln(x)$. Then $du = \frac{dx}{x}$ and we find that

$$\int_1^e \frac{\ln(x)}{x}\,dx = \int_0^1 u\,du = \left.\frac{u^2}{2}\right]_0^1 = 1/2.$$

Answer: $\int_1^e \frac{\ln(x)}{x}\,dx = \frac{1}{2}$.

4

We use the method of integrating factor. An antiderivative to $\sin(x)$ is $-\cos(x)$ so we multiply both sides of the equation by $e^{-\cos(x)}$. We get

$$(ye^{-\cos(x)})' = 1 \Leftrightarrow ye^{-\cos(x)} = x + C \Leftrightarrow y = xe^{\cos(x)} + Ce^{\cos(x)}.$$

We use the initial condition to determine the constant C. Since $y(0) = 1$ we find that

$$Ce^{\cos(0)} = 1 \Leftrightarrow Ce = 1 \Leftrightarrow C = e^{-1}.$$

Answer: The solution is $y = xe^{\cos(x)} + e^{\cos(x)-1}$.

5

We can use the ratio test. Set $a_n = \sqrt{n}e^{-n}x^n$ and consider

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n+1} \cdot e^{-n-1} \cdot |x|^{n+1}}{\sqrt{n}e^{-n}|x|^n} =$$
$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}}e^{-1}|x| = e^{-1}|x|.$$

By the ratio test the series converges if $e^{-1}|x| < 1$ and diverges if $e^{-1}|x| > 1$. The condition $e^{-1}|x| < 1$ is equivalent to $|x| < e$.

Answer: The radius of convergence is e.