Exercise assignment 1

Each exercise assignment has two parts. The first part consists of $3-5$ elementary problems for a maximum of 10 points from each assignment. For the second part consisting of problems in applications you are to choose *one* of three problems to solve. This part can give up to 5 points from each assignment. The first part consists of elementary questions to make sure that you have understood the basic material of the course while the second part consists of larger application examples.

Solutions can either be submitted through the Blackboard page or you can submit handwritten solutions in the envelope outside of room $U3-185$ before 23.59 on Sunday 7th of December.

Each exercise assignment can give a maximum of 15 points, to pass you will need at least 20 points total from both the assignments. If you do not get enough points from the assignments you will be given the opportunity to complement your solutions to reach a passing grade. These complements need to be submitted before 23.59 on the 11th of January.

1 Part 1

In the first part you are to solve and hand in solutions to the questions. You are allowed to use computer software to check your results, but your calculations as well as your result should be included in the answers for full points.

1.1

a) (1) Use the characteristic polynomial $\det(\mathsf{I}\lambda - \mathsf{A})$ to find the eigenvalues of A.

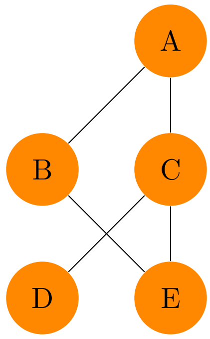
b) (1) Calculate the determinant of A

$$\mathsf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

1.2

a) (1) Define the degree matrix and the adjacency matrix for a undirected graph.

b) (1) Calculate the degree matrix and adjacency matrix for the graph below.



1.3

Consider the stochastic matrix M below.

$$\mathsf{M} = \begin{bmatrix} 0 & 0.75 & 0 & 0.25 \\ 0.25 & 0 & 0.75 & 0 \\ 0 & 0.25 & 0 & 0.75 \\ 0.75 & 0 & 0.25 & 0 \end{bmatrix}$$

a) (1) Write out the graph describing the transitions of this Markov chain.

b) (1) If we start in the first state, calculate the probability that we are in state 2 after 3 steps in the Markov chain described by M.

1.4

Consider matrix A below.

a) (1) Show that A is irreducible and primitive.

b) (1) Use the Perron-Frobenius theorem for non-negative irreducible matrices to find an estimate of the dominant eigenvalue of A (without calculating the eigenvalues).

$$\mathsf{A} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 1 \end{bmatrix}$$

1.5

We consider the matrices $\mathsf{P}, \mathsf{A}, \mathsf{L}, \mathsf{U}$ with LUP-factorization $\mathsf{PA} = \mathsf{LU}$.

$$\mathsf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \mathsf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -2 & 2 \\ 1 & -2 & 4 \end{bmatrix}, \; \mathsf{L} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \; \mathsf{U} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

a) (1) Show that P is an invertible, normal and unitary matrix.

b) (1) Use the LU-factorization above to solve $\mathsf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [-2\ 3\ 3]^\top$.

Part 2

In this second part you are to choose **ONE** example where you attempt to solve the questions presented. If you hand in answers to more than one choice you will get points corresponding to the choice which would give the least total points.

You are allowed to use computer software (and depending on which option you choose might be needed for some of the questions). If you are using a computer for some calculations you should set up the problem and present how it could theoretically be solved by hand. For example you could write "I solved the linear system $Ax = b$ using Matlabs "fsolve", another method would have been to use Gaussian elimination and solving the resulting triangular system".

Modelling student transitions

At another university college in Sweden, not MDH, they have the problem that not every student takes his or her degree in the planned time. Some students even fail to complete their education at all.

They have asked MDH if we could possibly model their problem for them. More specifically, we are interested in modelling the progress, or lack of progress, of students studying a two-year Master's programme at this college.

We can model the educational journey of a student using a Markov chains. In our model there are 4 states, representing whether a student is registered in year 1, year 2, has taken their exam or has dropped out.

They report that of the first year students, 80% continue to year 2, and the rest drops out. Of the year 2 students, 40% re-register in year 2, 20% take their exam and 40% drop out. The states of having dropped out or taken ones exam can be seen as absorbing states.

- (a) Write down the transition matrix for this model. Explain how you have numbered the states.
- (b) Are there any assumptions in the model that strike you as dubious?

To evaluate the educational quality, we are interested in some statistics.

- (c) What proportion of students that start the programme will drop out? (On average.)
- (d) What proportion of students will take their degree?
- (e) How many years will a student spend on average at the college before dropping out or taking their exam?

The answer to each question should be given a detailed motivation without reference to computer calculations.

PCA in finance

Financial data is often very large, making methods such as PCA useful in order to reduce the dimension of the data to manageble levels. In exercise we will consider a collection of k stocks and their price history over some time interval. Rather than doing further analysis on all k stocks our aim is to use PCA in order to reduce the size of this dataset without losing to much information.

In order to solve this problem you are likely to need to use computer software for the actual computations, whenever that is the case remember to describe what you are actually doing as well and not simply give for example the matlab command.

Consider the case where with 6 different stocks and their price history described in the matrix below, every row corresponds to one stock and the columns corresponds to the price at the end of each day over two weeks (excluding the weekend where we assume the market is closed).

$$\begin{bmatrix} 102 & 96 & 102 & 105 & 102 & 103 & 102 & 99 & 101 & 98 \\ 11 & 8 & 9 & 9 & 6 & 12 & 11 & 9 & 12 & 7 \\ 25 & 25 & 25 & 25 & 24 & 25 & 25 & 26 & 26 & 26 \\ 12 & 13 & 11 & 11 & 12 & 16 & 12 & 13 & 13 & 14 \\ 4 & 4 & 5 & 5 & 5 & 7 & 4 & 4 & 5 & 6 \\ 59 & 61 & 60 & 62 & 58 & 58 & 58 & 61 & 60 & 60 \end{bmatrix}$$

The data in this matrix can also be found in the matlab file "assignment1data.m" if you want to avoid writing out the whole matrix by hand.

- (a) Calculate the covariance matrix for this dataset. You will have to use computer software to do the actual computations, however every step you do should be described. (Make sure that you get a 6×6 covariance matrix).
- (b) Find the eigenvalues and eigenvectors of the covariance matrix and order the eigenvalues from the "largest" to the smallest (in terms of absolute value).
- (c) Choice a suitable number of principal components, motivate your choice. Your answer should include your eigenvalues and eigenvectors you have choosen to use.
- (d) Do steps a-c, but use the correlation matrix instead. Use the same criterium to choose the number of principal components.
- (e) If you where to choose one of these sets of principal components for further analysis of the data, which one would you choose? Give a clear motivation of your choice of principal components and what kind of analysis you are considering.

PCA in energy

The consumption of electrical energy for a household varies over time. It may depend on for example the time of day, time of week, time of month and time of the year. Different measures on this energy consumption can be related to each other. For example, the mean consumption in Watts between 6 pm and 7 pm may give clues for the mean consumption over the entire day.

Ten households where monitored for one day resulting in data in Figure 1.

$$A = \begin{bmatrix} 100 & 200 \\ 300 & 200 \\ 300 & 400 \\ 700 & 400 \\ 600 & 600 \\ 900 & 600 \\ 1000 & 400 \\ 1100 & 800 \\ 1300 & 500 \\ 1600 & 700 \end{bmatrix}$$

Figure 1: Each row in this matrix is for one household and on the form (c_{67}, c_{24}) where c_{67} is the mean consumption between 6 pm and 7 pm and c_{24} is the mean consumption over the entire day.

- (a) Plot the set of points with coordinates defined by the rows of the matrix A. Can you see any pattern? Construct a new matrix B by centering the data in Figure 1 by subtracting the mean values of c_{67} and c_{24} from the respective columns.
- (b) Construct the 2×2 covariance matrix C for the matrix B.
- (c) Calculate the eigenvalues and the corresponding eigenvectors of the covariance matrix C.
- (d) Select one suitable eigenvector from the two calculated in (c) and project the data on the subspace defined by this eigenvector. The resulting matrix D should be of the same dimensions as A and B. Now add the suitable means to each row in D to get the matrix E in the original coordinate system as that for the original matrix A. Plot this new set of points contained in E.
- (e) Discuss what has changed as we went from the set of points in A to the set of points in E. If we assume that there should be a linear relationship between c_{67} and c_{24} what would that be according to the points in E? How can the equation for this line be computed directly from the means of c_{67} and c_{24} and the chosen eigenvector?

Modelling supervision of a production line.

Cosider a machine (or series of machines) producing some commodity. In theory the machine should be fully autonomous, however sometimes problems occur and the machine need to be stopped temporarily. To this end the company makes sure that there is always one person supervising the machine as long as it's running.

Usually whenever a problem occur the supervisor notices the problem, stops the machine for a short while to fix the problem and then resumes production. However if the supervisor does not notice the problem in time the machine will have to undergo repairs before it can run again.

The company want to know if it wouldd be more profitable to hire more (or less) supervisors of the machine and to that end want to model the production of the machine. We will choose to do this using a discrete Markov process.

The machine is assumed to have only two states "running" and "under repairs". In every step of the process (representing 1 hour) there is a 0.3 probability that an error occurs which if not noticed in time will result in the machine having to be repaired (transition to "under repairs" state). For every hour the machine is under repairs we assume there is a 0.5 probability that the machine will be fixed and able to run for the next hour. This can be described using the following Markov chain for the machine with no supervisor (i.e. errors are never noticed in time).

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}$$

We model the supervisors using another Markov chain with two states, one "alert" state and one "tired" state. In the alert state the supervisor notices an error in time with a probability 0.8, however in the "tired" state it will only notice it with a probability 0.5. Whenever an error was noticed (in time) last period or the machine just finished repairs the supervisor will be in the "alert" state for the next period. However if no error occured last period the supervisor enters the "tired" state and stays there untill an error occurs.

- (a) Modify the model for the machine with no supervisor such that it includes 1 supervisor using a 3-state markov chain representing the states "running, alert", "running, tired", "under repairs". Write out the stochastic matrix for this new Markov chain.
- (b) What is the expected number of steps until the machine first enters the "under repairs" state if we start in the "running, alert" state?
  Hint: Consider the "under repairs" state as an absorbing state.
- (c) Find the stationary distribution of our Markov chain, what is the proportion of time the machine is in the "under repairs" state if we look over a long time period?

Every hour the Markov chain spends in the "under repairs" state it incurs a cost c_r, similarly every time period during which the machine is running gives a reward r. In every step there is also a cost c_p in order to pay for the supervisor of the machine, this cost is payed even while the machine is undergoing repairs.

- (d) Let $c_r = -20$, $r = 100$, $c_p = -10$, what is the long term expected reward of one step in the Markov chain? (average profit/hour).

- (e) Consider the same machine but with two independent supervisors (assumed to both always be in the same state and swapping to alert whenever one of them notices an error or the machine just finished repairs). What it the new long term expected reward of the Markov chain considering the new probability to notice errors as well as double personel cost (c_p). Would it be good for thecompany to assign two supervisors to the machine rather than one?