A short example calculating eigenvalues and eigenvectors of a matrix

We want to calculate the eigenvalues and the eigenvectors of matrix A:

$$\mathsf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

We start by using the Characteristic polynomial to find the eigenvectors:

$$\det(\lambda \mathsf{I} - \mathsf{A}) = \det \begin{bmatrix} \lambda - 2 & 1 & 0 \\ -1 & \lambda + 1 & -1 \\ 1 & 1 & \lambda - 1 \end{bmatrix}$$

(Along the first row)

$$= (\lambda - 2) \det \begin{bmatrix} \lambda + 1 & -1 \\ 1 & \lambda - 1 \end{bmatrix} - (-1) \det \begin{bmatrix} -1 & -1 \\ 1 & \lambda - 1 \end{bmatrix}$$

$$= (\lambda - 2)\left((\lambda - 1)(\lambda + 1) + 1\right) + (\lambda - 1) - 1 = (\lambda - 2)(\lambda^2 + 1)$$

Setting the determinant to zero gives:

$$(\lambda - 2)(\lambda^2 + 1) = 0 \Rightarrow \lambda = 2, \pm i$$

To fing the eigenvector to $\lambda = 2$ we set upp the resulting linear system:

$$\mathsf{A}\mathbf{x} = \lambda \mathbf{x} = 2\mathbf{x} \Leftrightarrow (2\mathsf{I} - \mathsf{A})\mathbf{x} = 0$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

We solve this using Gaussian elimination. We start by exchaning first and last row to get the zero in the bottom left corner (we could of course instead reduce to a lower triangular matrix rather than upper as we do here):

$$\begin{bmatrix} -1 & 3 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Next we add row 1 to row 2:

$$\begin{bmatrix} -1 & 3 & -1 \\ 0 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

And last we add 1/4 of row 2 to row 3:

$$\begin{bmatrix} -1 & 3 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

We notice that we get a zero row (which we can expect whenever we try and find an eigenvector since we set the determinant to zero).

Looking at the 2nd equation we get $4x_2 = 0 \Rightarrow x_2 = 0$ which when put in the first equation gives:

$$-x_1 - x_3 = 0 \Rightarrow x_3 = -x_1$$

and we get eigenvector

$$\mathbf{x} = k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Where k is any non-zero constant.
Doing the same for $\lambda = i$ gives

$$\begin{bmatrix} i-2 & 1 & 0 \\ -1 & i+1 & -1 \\ 1 & 1 & i-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Multiplying adding row 2 to row 3 as well as multiplying row 1 with $(i-1)^{-1}$ and ad it to row 2 gives:

$$\begin{bmatrix} 1 & (i-2)^{-1} & 0 \\ 0 & i+1+(i-2)^{-1} & -1 \\ 0 & i+2 & i-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Putting row 2 on the same denominator and multiplying with the denominator gives:

$$\begin{bmatrix} 1 & (i-2)^{-1} & 0 \\ 0 & -(i+2) & -(i-2) \\ 0 & i+2 & i-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Last row is a linear combination of the 2nd and we end up with:

$$\begin{bmatrix} 1 & (i-2)^{-1} & 0 \\ 0 & 2+i & -2+i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The 2nd row gives $(2+i)x_2 = (2-i)x_3 \Rightarrow x_3 = \dfrac{2+i}{2-i}x_2$ and the first row gives $x_1 = 1/(-2+i)x_2$ giving eigenvector:

$$\mathbf{x} = k \begin{bmatrix} \dfrac{1}{-2+i} \\ 1 \\ \dfrac{2+i}{2-i} \end{bmatrix}$$

Doing the same for the 3rd eigenvalue result in the same eigenvector as for $\lambda = i$.