

# Applied Matrix Analysis, MAA704

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Today's lecture:

- ▶ Course overview.
- ▶ Repetition of matrices and elementary operations.
- ▶ Repetition of solvability of linear equation systems, eigenvectors and eigenvalues.

# Course overview

- ▶ 11 lectures of 3 – 4 hours.
- ▶ Examination consists of 2 exercise assignments and one project with presentation.

## lecture 1

Linear  
equations  
systems

Rank

Determinants

Inverse

Eigenvectors  
and  
eigenvalues

# Changes from previous years

- ▶ Greater focus on multivariate analysis techniques.
- ▶ Mainly in order to accomodate for the changes the course compendium will be updated during the course.

# Repetition matrices

A matrix is a rectangular array of numbers, symbols or expressions. Some example of matrices can be seen below:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1+i & -2 \\ 0 & 1-i \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -2 \\ -2 & 3 \\ 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \end{bmatrix}$$

Note that matrices need not be square as in C and D.

# Repetition matrices: notation

We take a look at matrix  $C = \begin{bmatrix} 1 & -2 \\ -2 & 3 \\ 3 & 1 \end{bmatrix}$

- ▶ We denote every number or rather element of  $A$  as  $a_{i,j}$ , where  $i$  is it's row number and  $j$  is it's column number. What is  $c_{3,1}$  of  $C$  above?
- ▶ The size of a matrix is the number of rows and columns (or the index of the bottom right element): We say that  $C$  above is a  $3 \times 2$  matrix.
- ▶ A matrix with only one row or column is called a row vector or column vector respectively.
- ▶ The diagonal of a matrix is the elements diagonally from the top left corner towards the bottom left ( $a_{1,1}, a_{2,2}, \dots, a_{n,n}$ ). A matrix where all elements not on the diagonal is zero is called a diagonal matrix.

# Repetition matrices: addition and subtraction

Add every element in the first matrix with correspondign element in the second matrix. We demand that both matrices have the same size.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+1 & 0-2 \\ 0+2 & 2-1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$$

# Repetition matrices: multiplication

Given the two matrices A and B we get the product AB as the matrix whose elements  $e_{i,j}$  are found by multiplying the elements in row  $i$  of A with the elements of column  $j$  of B and adding the results.

$$\begin{bmatrix} 3 & 2 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 10 & 7 & 14 \\ 5 & 9 & 16 & 24 \\ 1 & 4 & 8 & 9 \\ 0 & 1 & 5 & 10 \end{bmatrix}$$

For the colored element we have:  $1 \cdot 3 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 = 5$



# Repetition matrices: multiplication

We note that we need the number of columns in  $A$  to be the same as the number of rows in  $B$  but  $A$  can have any number of rows and  $B$  can have any number of columns. Also we have:

- ▶ Generally  $AB \neq BA$ , why?
- ▶ The size of  $AB$  is the number of rows of  $A$  times the number of columns of  $B$ .
- ▶ The Identity matrix  $I$  is the matrix with ones on the diagonal and zero elsewhere. For the Identity matrix we have:  $AI = IA = A$ .

# Repetition matrices: multiplication

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Multiplying a scalar with a matrix is done by multiplying the scalar with every element of the matrix.

$$3 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 0 \\ 3 \cdot 0 & 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

# Linear equation systems and matrices

- ▶ Linear equation system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots x_1 \quad \quad \quad \vdots x_2 \quad \quad \quad \ddots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

- ▶ Represented with a matrix and column vectors

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

# Solving a linear equation system

- ▶ Gaussian elimination.
- ▶ Elementary row operations (adding and multiplying rows).
- ▶ Find pivot elements, change matrix to diagonal/echelon row form.

$$\begin{bmatrix} 0 & \circledast & * & * & * & * & * & * \\ 0 & 0 & 0 & \circledast & * & * & * & * \\ 0 & 0 & 0 & 0 & \circledast & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \circledast & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \circledast \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# When can a linear system be solved?

From previous courses:

## Theorem

*The following statements are equivalent for  $A \in \mathcal{M}_{n \times n}(\mathbf{K})$ :*

- 1.  $AX = B$  has a unique solution for all  $B \in \mathcal{M}_{n \times m}$  where  $X \in \mathcal{M}_{n \times m}$ .*
- 2.  $AX = 0$  only has the trivial solution ( $X = 0$ ).*
- 3.  $A$  has linearly independent row/column vectors.*
- 4.  $A$  is invertible/non-degenerate/non-singular.*
- 5.  $\det(A) \neq 0$ .*
- 6.  $A$  has maximum rank ( $\text{rank}(A) = n$ ).*
- 7.  $A$ 's row/column space has dimension  $n$ .*
- 8.  $A$ 's null space has dimension  $0$ .*
- 9.  $A$ 's row/column vectors span  $\mathbf{K}$*

## Definition (1)

For  $A \in \mathcal{M}_{n \times n}(\mathbf{K})$  the  $\text{rank}(A)$  is the number of linearly independent rows or columns. To see that the number of linearly independent rows is the same as the number of linearly independent columns, draw a matrix in echelon form.

## Definition (2)

For  $A \in \mathcal{M}_{n \times n}(\mathbf{K})$  the  $\text{rank}(A)$  is dimension of the column space of  $A$ .

# What do we already know about determinants?

Let  $A, B \in \mathcal{M}_{n \times n}(\mathbb{K})$

- ▶ Know how to calculate it for  $1 \times 1$ ,  $2 \times 2$  and  $3 \times 3$  matrices.
- ▶ Know how to calculate it for triangular matrices and matrices with linearly dependent rows.
- ▶  $\det A^T = \det A$
- ▶  $\det AB = \det A \det B$  but  $\det(A + B) \neq \det(A) + \det(B)$ .
- ▶  $\det cA = c^n \det A$
- ▶ ... plus a few more things.

# Calculating determinants using cofactors

## Theorem

$$\det(A) = \sum_{k=1}^n a_{ki} A_{ki}$$

where  $A_{ki}$  is the  $ki$ -cofactor of  $A$ .

The  $ki$ -cofactor is  $(-1)^{i+k}$  times the determinant of  $A$  after we have removed row  $k$  and column  $i$ .



## Definition

The inverse of a matrix  $A \in \mathcal{M}_{n \times n}$  is a matrix  $X \in \mathcal{M}_{n \times n}$  such that

$$AX = XA = I$$

The inverse is denoted  $X = A^{-1}$ .

- ▶ If  $AX = B$  then  $A^{-1}AX = A^{-1}B \Leftrightarrow IX = X = A^{-1}B$  and thus we can solve linear equation systems if we know the inverse.

# What do we know about inverses?

- ▶  $(A^{-1})^{-1} = A$
- ▶  $(A + B)^{-1} \neq A^{-1} + B^{-1}$
- ▶  $(AB)^{-1} = B^{-1}A^{-1}$
- ▶  $(aB)^{-1} = \frac{1}{a}B^{-1}$
- ▶  $(A^{-1})^T = (A^T)^{-1}$

# Calculate the inverse

- ▶ Use gaussian elimination or the adjugate formula

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

where  $\text{adj}(A)$  is the adjugate matrix.

## Definition

The adjugate matrix  $\text{adj}(A)$  of a matrix is the transpose of the cofactor matrix where we have  $(\text{adj}(A))_{ik} = A_{ki}$ .

# Other kinds of inverses

- ▶ Left and right inverses

$$BA = I, AB = I$$

- ▶ Pseudoinverses

$$ABA = A \text{ and } BAB = B$$

- ▶ These kinds inverses can exist for non-square matrices,  
 $A \in \mathcal{M}_{m \times n}, B \in \mathcal{M}_{n \times m}$

# Eigenvalues and eigenvectors

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lecture 1

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## Definition

If  $A\mathbf{v} = \lambda\mathbf{v}$  where  $A \in \mathcal{M}_{n \times n}(\mathbf{K})$ ,  $\lambda \in \mathbf{K}$ ,  $\mathbf{v} \in \mathcal{M}_{n \times 1}(\mathbf{K})$  then  $\lambda$  is an *eigenvalue* of  $A$  and  $\mathbf{v}$  is an *eigenvector* if  $\mathbf{v} \neq \mathbf{0}$ .

# What do we know about eigenvalues and eigenvectors?

- ▶ Can have several different eigenvalues and eigenvectors.
- ▶ Different eigenvectors can have the same eigenvalue.
- ▶ Can find the eigenvalues by solving  $\det(A - \lambda I) = 0$  and the eigenvectors by solving  $(A - \lambda I)\mathbf{v} = 0$ .

# Spectrum of a matrix

- ▶ Set of eigenvalues for a matrix  $A$  is called the *spectrum* of  $A$  and is denoted by  $\text{Sp}(A)$ .
- ▶ The absolute value of the largest eigenvalue is called the *spectral radius*

$$\rho(A) = \max_{\lambda \in \text{Sp}(A)} |\lambda|$$

# Finding the eigenvalues

- ▶  $A\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow (A - \lambda I)\mathbf{v} = 0$  if  $\mathbf{v} \neq 0$  then  $\det(A - \lambda I) = 0$ .
- ▶ Many different methods, for example Gaussian elimination.
- ▶ Can find the eigenvalues by solving  $\det(A - \lambda I) = 0$  which is an  $n$ th degree polynomial in  $\lambda$ .
- ▶  $p_A(\lambda) = \det(A - \lambda I)$  is called the *characteristic polynomial* and  $p_A(\lambda) = 0$  is called the *characteristic equation*.



# Important results with eigenvalues

- ▶  $\operatorname{tr}(A) = \sum_{i=1}^n a_{i,i} = \sum_{i=1}^n \lambda_i$
- ▶  $\det(A) = \prod_{i=1}^n \lambda_i$
- ▶ If  $A \in \mathcal{M}_{n \times n}(\mathbf{K})$  have  $n$  different eigenvalues then  $A$  is diagonalizable.