Course information for Discrete Mathematics, MMA122

Examiner: Kimmo Eriksson, UKK, Mälardalen University

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Innehåll

1 Take the chance to study!

Lectures and lessons in this course sum up to about 50 hours. The course counts as half-time studies for 10 weeks, which means that you are expected to study an additional 150 hours on your own. Please do! Spend some time every day reading in the textbook and solving exercises.

2 Aims and ideas behind the course

The aim of the course is to introduce the basic concepts and methods in discrete mathematics, and to give improved proficiency in mathematical modeling, problem solving and reasoning, as a basis for further studies in mathematics and computer science. At the end of the course the student is expected to be able to

- explain, in a way adapted to the mathematical level of the reader/listener, the concepts presented in this course,
- describe some application of each of the subareas of the course content,
- use properly the set algebraic operations and set up models to solve problems by set algebraic means, and describe the relation between propositional logic and boolean algebra,
- formulate and interpret statements written in the notation of predicate logic,
- give an account of the concepts of prime numbers and divisors, and apply Euclid's algorithm to problems such as linear modular equations, - prove theorems by induction, and solve problems that rely on recursion,
- describe and apply the fundamental methods and principles of combinatorics and probability theory,
- use basic graph theoretic terminology and set up models to solve problems by graph theoretic means,
- construct and interpret automata, and describe the relation between automata and regular languages.

The idea behind the course is to provide some freedom of choice in assignments and level of ambition. It is the student's responsibility to use this freedom of choice in a wise way.

3 Literature and lecture plan

The literature is the textbook *Discrete mathematics and discrete models,* by Kimmo Eriksson and Hillevi Gavel. The lectures follow the chapters in the textbook.

Lec1 Set theory

Lec2 Arithmetic

Lec3 Recursion and proof by induction

Lec4 Combinatorics and probability

Lec5 Graphs

Lec6 Logic and boolean algebra

Lec7 Artificial languages and finite automata

Lec8 Preparation for oral exam

4 Lectures vs. lessons

The point of the lectures is to introduce the subject matter in order to guide and inspire further reading in the textbook. Attendance at lectures is recommended but not mandatory. During lessons, you are divided into groups of no more than five people. Groups are formed at the first lesson. During the lessons, the teacher demonstrates solutions to exercises and conducts examination of weekly assignments, see the schedule. Attendance at examination sessions is mandatory. If you miss an examination, you must discuss with the teacher how to make it up at a later lesson.

5 Teachers

Lectures:

- *Kimmo Eriksson*, professor, kimmo.eriksson@mdh.se

Lessons:

- Mahamadi Ouoba (A-ANFI), junior lecturer, mahamadi.ouoba@mdh.se
- Ying Ni (B), senior lecturer, ying.ni@mdh.se

6 Examination

Examination consists of Exercises (3 credits) and Oral examination (4.5 credits).

6.1 Exercises (3 credits)

Exercises consist of seven *weekly assignments* and one individual report (the *I-report*). The 3 credits are LADOK reported when the student has passed all the exercises. The mark on the I-report (3 or 5) determines the mark reported in LADOK.

6.1.1 Weekly assignments

Described in detail in **appendix 1.** In brief, groups choose one assignment per week and come up with a joint solution. At the examination, the group presents their solution to another group who will have worked on a different assignment; they must also listen to that group's presentation of their solution. Both groups then present these solutions to the teacher. If you miss an examination, you must discuss with the teacher how to make it up at a later lesson. The last chance to make up for missed examinations is at the last scheduled lesson.

Every member of the group must also hand in an individually handwritten solution to the teacher. The written solution must satisfy the following criteria:

- Start by formulating the problem.
- Write text to explain all the steps in the solution. This includes (a) spelling out all assumptions, (b) refering back to assumptions when they are used, (c) specifying which formulas and theorems are used in the solution, (d) defining all symbols before they are used.
- Be careful with grammar and spelling.
- Phrase your answer to the solution in a complete sentence that stands for itself.

6.1.2 I-report

Described in detail in **appendix 2.** In brief, every student chooses an individual assignment among a number of rather comprehensive problems. The solution must be documented in a report written using a word processor. All reports are checked for plagiarism through the Urkund system. I-reports are marked 3 or 5 (unless failed).

6.2 Oral examination (4.5 credits)

Described in detail in **appendix 3.** Marked 3 or 5 (unless failed). Sign up for the oral exam on Blackboard.

6.3 Retaking examinations

Weekly assignments cannot be saved until the next time the course is given. In other words, if you do not pass all weekly assignments during the course, you need to retake the course and do a full set of weekly assignments. Similarly, if your I-report does not pass during the course you must retake the course.

If you fail the oral exam you can retake it at a later date. Oral examinations are given three times a year: in mid-autumn, in mid-spring, and at the end of the spring term. There is no oral exam given in August.

6.4 Final mark

The final mark (3, 4 or 5) is given by the average of the two marks given on the oral exam and the I-report.

7 Environmental aspects

There are no direct environmental aspects to this course.

8 Appendix 1: Weekly assignments

8.1 Choice of assignments

Assignments come under four headings: *Problem solving, Modeling, Reasoning,* and *Calculation*. At the end of the course, your group must have chosen at least one assignment under each heading. Every week you have a choice between two headings, defined by the parity of your group number (odd or even). Group numbers are given out at the first lesson.

8.2 Presentation of solutions

Presentation of solutions involve an oral part and a written part. The oral part is group-based and consists of the following steps.

(a) The group selects one group member's written solution to be used for the oral presentation.

(b) The teacher matches up groups that have solved different assignments. The groups present their solutions to each other until both groups have completely understood the other group's solution (or pointed out a weakness in it).

(c) Each group presents the other group's solution to the teacher! This may result in the teacher pointing out weaknesses that must be addressed.

The written part is individual-based and consists of every student handing in an individually hand-written solution to the teacher. When the oral presentation is deemed acceptable, the students have the opportunity to make corrections to their written solutions before handing them in. The written solution must satisfy the following criteria and the teacher will return for revision any hand-ins that do not satisfy these criteria:

- Start by formulating the problem.
- Write text to explain all the steps in the solution. This includes (a) spelling out all assumptions, (b) refering back to assumptions when they are used, (c) specifying which formulas and theorems are used in the solution, (d) defining all symbols before they are used.
- Be careful with grammar and spelling.
- Phrase your answer to the solution in a complete sentence that stands for itself.

The aim of this combination of oral and written presentation is to improve students' skills at presenting and discussing mathematics in an intelligible and correct way. If you miss an examination, you must discuss with the teacher how to make it up at a later lesson. The last chance to make up for missed examinations is at the last scheduled lesson.

Ch. 2

Problem solving (ONLY FOR ODD-NUMBERED GROUPS)

A Swedish and a Finnish university has a joint research project concerning the cultural exchange between Finland and the Finnish part of Sweden. The project participants (both academics and field workers) should now come together for a three-day conference. The organizers shall determine which language to use. From a questionnaire sent out in advance the following is known:

- A total of 57 people will be participating.
- A total of 42 participants are able to speak Finnish.
- A total of 38 participants are able to speak Swedish.
- A total of 35 participants are able to speak English.
- 23 participants are able to speak both Swedish and Finnish.
- 20 participants are able to speak both Finnish and English.
- 27 participants are able to speak both Swedish and English.

Unfortunately, it is not known how many are able to speak all three languages. The organizers now want to find out a couple of things:

1. Do they have to print the conference information in all three languages or can they get away with using only one or two in order for all participants to be able to understand the information? Hint! Use the identity in Exercise 2.28 (2.19 in the old edition) to figure out how many people are able to speak all three languages. Draw a Venn diagram with three sets intersecting and calculate how many people are in each cell of the Venn diagram.

2. Each of the three days of the conference will be in a different language (i.e., Swedish one day, English one day, Finnish one day). Can the organizers get away with using less than two interpreters each day?

Modeling (ONLY FOR ODD-NUMBERED GROUPS)

The University will develop guidelines by which you should be able to answer issues such as:

- When is it OK for a student to replace the obligatory but extinct course A by the currently existing course B to obtain their degree?
- When is it not OK to use both courses A and B to obtain a degree?
- Under what conditions can course A and course B together be used to replace course C as entry requirements?

Model the problem using set theory by representing a course by the set of all its contents. (E.g., the course in Discrete mathematics would be represented by the set {set theory, arithemetic, recursion, etc.}. Using this model, suggest how the above questions might be answered in the form of set-theoretical assertions.

Reasoning (ONLY FOR EVEN-NUMBERED GROUPS)

Explain why the following generalized distributive law holds true:

$$A \cup (B \cap C \cap D) = (A \cup B) \cap (A \cup C) \cap (A \cup D)$$

Your argument should take an arbitrary element from the set described in the left-hand side of the law and explain why this element also must belong to the set described in the right-hand side, and vice versa. For unions, this means that a case distinction must be made. Your explanation might begin as follows: "There are two cases for an element in the left-hand set: either the element belongs to the set A or it simultaneously belongs to the three sets B, C and D. We begin with the case that the element belongs to A and show that in this case the element must also belong to the right-hand set. What we need to show then is that the element belongs simultaneously to $A \cup B$ och $A \cup C$ och $A \cup D$. Well, given that the element lies in the set A, it is self-evident that it also belongs to these three sets as they are unions of A with other sets."

Continue with the second case in this direction. Then do the reasoning in the opposite direction, showing that an element that belongs to the right-hand set also must belong to the left-hand set.

Calculation (ONLY FOR EVEN-NUMBERED GROUPS)

Show that the following two set expressions:

$$A \cup \big(B \cap (A \cup C)\big) \quad \text{and} \quad \big(A \cup (A \cup B^c)^c\big) \cap (A \cup C)$$

actually describe the same set, by using the various laws for calculating with sets. Hint! Show that both expressions, through a sequence of rewriting steps using the laws, can be simplified to:

$$A \cup \big(B \cap C)$$

The calculation must be neatly arranged and easy to follow; each line of the solution must refer to a named law to allow rewriting the expression one step. For example, the left-hand expression can in two steps be rewritten as follows:

$$A \cup \big(B \cap (A \cup C)\big) = A \cup \big((B \cap A) \cup (B \cap C)\big) \quad \text{[according to the distributive law]}$$

$$= \big(A \cup (B \cap A)\big) \cup (B \cap C) \quad \text{[according to the associative law]}$$

Proceed in the same manner!

Ch. 3

Problem solving (ONLY FOR EVEN-NUMBERED GROUPS)

The way to solve diophantic linear equations is presented in the book. But what about *systems* of diophantic linear equation? Figure out a way to solve the system $[6x + 20y + z = 200; x + y + z = 100]$, given the constraint that all three variables belong to $\mathbb{N}$. (Hint: Start by eliminating one variable; then solve the remaining equation; finally, include the eliminated variable in the solution.)

Modeling (ONLY FOR EVEN-NUMBERED GROUPS)

In an army base, a machine for encryption of secret messages uses three cogwheels[1]. The wheels have 17, 39 and 64 cogs, respectively. For each character that is sent, all the wheels are turned one step forward each. How the character is encrypted depends on how the wheels are positioned. The receiver of the message has an identical machine operating at the other end of the line, decrypting according to the same principle. (This machine is authentic.)

At the end of the day, the colonel comes in. He is curious about how many characters have been sent this day. Unfortunately, this has not been recorded. Nevertheless, one soldier promises to find out the answer from the position of the cogwheels. At the beginning of each day, all cogwheels are placed at "position zero". When the colonel arrived, the position of the first wheel was 3 cogs forward from position zero, the second wheel 5 cogs forward from position zero, and the third wheel 4 cogs forward from position zero. Introduce a variable for the number of characters sent this day, and express the system of modular equations that this variable must satisfy. (You don't need to solve this system of equations, only write it up and explain why it describes the information given above. If you are interested in how to actually solve such systems of modular equations, search for "Chinese Remainder Theorem" on the internet.)

Reasoning (ONLY FOR ODD-NUMBERED GROUPS)

Explain why the *commutative* and *associative* laws for addition and multiplication hold in modular arithmetics.

It has to be crystal clear to the reader of your solution how the reasoning goes, and the argument must be watertight. Hint: The solution ought to start: "The commutative law for addition in modular arithmetics says that $x+y \equiv y+x \pmod{n}$, which by definition is equivalent to $x+y = y+x+kn$ for some integer k." Then show that there is a value of k for which the latter equality holds. Then do an analogous proof for multiplication and for the associative law.

Calculation (ONLY FOR ODD-NUMBERED GROUPS)

If an integer ends on 0 or 5, then it is divisible by 5. If it ends on 0, 2, 4, 6 or 8, then it is divisible by 2. It is divisible by 3 if the sum of its digits is

[1] kugghjul

divisible by three.

Using these simple rules, determine the prime-factorings of 2295 and 2040. Then use these results to compute the prime-factorings of the following three numbers: $2295 \cdot 2040$, gcd(2295,2040) and lcm(2295,2040). Finally, demonstrate the relationship between these three numbers.

Ch. 4

Problem solving (ONLY FOR ODD-NUMBERED GROUPS)

If you build a triangular pyramid of marbles in three layers, there is one marble at the top, three marbles in the second layer, and six marbles in the third layer, a total of 10 marbles. Let P_n denote the total number of marbles in a triangular pyramid with n layers. Let L_n denote the number of marbles in the nth layer. It then holds that$L_1 = 1$,$L_2 = 3$, and $L_3 = 6$, and that $P_1 = 1$,$P_2 = 4$ and $P = 10$. (a) Set up and explain a recursion for L_n. (b) Prove that the formula $L_n = n(n+1)/2$ holds for all $n{>}0$. (c) Set up and explain a recursion for P_n. (d) Prove that the formula $P_n = n(n+1)(n+2)/6$ holds for all $n{>}0$.

Modeling (ONLY FOR ODD-NUMBERED GROUPS)

Roger the farmer grows a number of different crops (e.g., forage maize, rape, wheat, and grass). For each crop he grows a certain number of hectares, each crop gives a certain yield measured in tons per hectare, and each crop fetches a certain price measured in pounds per ton. Finally, each crop costs him a certain amount to plant, measured in pounds per hectare. Introduce a symbol for the number of crops and introduce symbols for each of the four numbers for each crop (distingusihing between crops by an index). Express the total profit using the sum symbol!

Reasoning (ONLY FOR EVEN-NUMBERED GROUPS)

Let $P(n)$ denote some mathematical proposition about the number n. ("$n(n+2)$ is an even number" is an example of what such a proposition might look like. "$2n > n$" is another example. It could be any mathematical statement involving the number n.) First, let's assume that somehow you know that $P(1)$ is a true statement. (To illustrate: $P(1)$ is not true for the first example above, as $1 \cdot (1+2) = 3$ is not an even number. However, $P(1)$ is true for the second example, as $2 \cdot 1 = 2 > 1$.) Second, let's assume that somehow you also know that *if* the statement$P(p)$ would be true for some given value$p$ then the statement $P(p+1)$ *will* also be true. Now explain to a skeptical person why this allows you to say that $P(2)$ is true, that $P(3)$ is true, that $P(4)$ is true, and in fact that $P(n)$ is true for every $n \geq 1$.

Your explanation should start as follows: ”We know by the fist assumption that $P(1)$ is true. By putting $p = 1$ in the second assumption we then know that ...” Complete the reasoning up to$P(4)$ and then explain how the same kind of reasoning can be used to reach any $n \geq 1$.

Calculation (ONLY FOR EVEN-NUMBERED GROUPS)

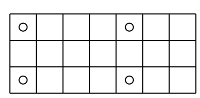
Prove using induction that the Fibonacci recursion ($f_{n+1} = f_n + f_{n-1}$;$f_0 = 0$; $f_1 = 1$) has the solution

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Ch. 5

Problem solving (ONLY FOR EVEN-NUMBERED GROUPS)

In an office occupied by 21 desks organized in a 3×7 rectangular pattern, there is one person at each desk.



The task is to prove that whoever these persons are, it is unavoidable that somewhere in the pattern there is a single-sex rectangle in the sense of four people of the same sex who sit in the corners of a rectangle (for example at the tables marked with rings in the figure). First, prove that in each column there must be two people of the same sex (using the pigeon hole principle). Then prove that for a larger office of size 3×9 there must be a single-sex rectangle, by proving that in such an office there must be two columns with identical gender placement (again using the pigeon hole principle). For a 3×7 sized office, you must analyze two cases. Start with the case that there is some column in which all three persons have the same gender, and show that then there is certainly a single-sex rectangle (again using the pigeon hole principle). Then proceed with the case that there is no column in which all three have the same sex (count the number of different possible columns that do not have all three of the same sex), and show that then too there is certainly a single-sex rectangle (again using the pigeon hole principle). Conclude that every 3×7 sized office must have a single-sex rectangle. Finally give an example of a 3×6 sized office that has no single-sex rectangle (which shows that the critical width is 7).

Modeling (ONLY FOR EVEN-NUMBERED GROUPS)

A simple model of how a flu virus spreads looks like this: Anyone who gets the virus has 1 week's incubation time before the symptoms arise. During that week the person will meet a number of people and for every person there is a certain probability that the virus is transmitted. When the first week is over the person will stay at home and not transmit the virus any further. Let $t = 0$ denote the time when the first person gets contagious with a new type of flu. Introduce variables for

- the probability that the virus is transmitted,
- the expected number of persons who are infectious week t, and
- the number of people that the average person typically meets in a week.

Then set up a recursion that describes the number of people expected to be infectious in week t. Use the recursion to calculate the number of people expected to be infectious week 1, 2, 3, etc. up to 12 (regular length of flu season), given that the probability of transmission is 10 percent and that the number of people the average person typically meets in a week is (a) 10, (b) 100. What conclusion can be drawn about what is required for an infection to become an epidemic? Compare also with the size of the entire global population. Is any result unreasonable? What does the model not take into account? How can the model be modified to avoid this kind of unreasonable results?

Reasoning (ONLY FOR ODD-NUMBERED GROUPS)

If you have two different messages you want to be able to send, you can encode them in binary as a single bit (e.g., "yes" = 1 and "no" = 0). One problem with this is that occasionally there are transmission errors (when a one comes across as a zero or vice versa), and when this happens the message will be misunderstood. To reduce the risk of misunderstandings one can instead use a binary code that corrects one error. This means that each message is encoded as a longer string of bits, so that if one bit goes wrong this can be detected and corrected. An example of such a code with two codewords for different messages is $\{111, 000\}$. If there are is an error when you send 111 so that, for example, it comes across as 101, you know it is very likely the actual message was 111. (Had it been 000 that arrived as 101, two errors in transmission must have occurred, which is much less likely.) If you want to send more than two messages and still be able to correct a transmission you must use longer codewords than three bits. Here is an example of a code with three codewords of five bits in length: $\{11111, 00011, 01100\}$.

(a) Show that this code really can correct one error! To do this, for each codeword form the set of all strings that can arise through the occurrence of at most one error. Verify that these sets are disjoint.

(b) Show that the number of binary strings of length L bits is 2^L.

(c) Show that for a codeword of length L there are $(1 + L)$ strings that may arise through the occurrence of at most one error.

(d) Show that if you want to send N different messages with a co- de that can correct one error, the length L of codewords must satisfy the condition$2^L \geq N \cdot (1 + L)$. Hint: Consider task (a), where you had $N = 3$ and $L = 5$. Then use (b) and (c).

Calculation (ONLY FOR ODD-NUMBERED GROUPS)

A circuit is made up of a series of parallel components. The circuit works if there is at least one functioning component in every parallel connection. For each component below we give the probability for it to be broken. What is the probability that the circuit will work in the following three cases?

(a) Series connection of two components with $P = 0.1$ resp 0.2. .

(b) Parallel connection of two components with $P = x$ and y, respectively.

(c) Series connection of two parallel connections, one with two components with $P = x$ and 0.5, the other with three components with$P = 0.5$ for each one. Determine the value of x for which the probability that the circuit will function is exactly one half. How many different combinations of broken and whole components produce a working circuit?

Ch. 6

Problem solving (ONLY FOR ODD-NUMBERED GROUPS)

Some sociology students have mapped out the relationships between the people working in different departments of the university. To preserve anonymity, the students used codes for the interviewees. Among the things they charted are which people within a department that under no circumstances are willing to cooperate. Some other students in the course are now compiling data, and they have become curious about whether two data sheets come from the same department or not. The data sheets consist of two lists of pairs, where each pair describes two people who cannot cooperate:

$$(a,b),\quad (a,c),\quad (a,e),\quad (a,f),\quad (a,h),\quad (b,c),\quad (c,d),\quad (c,f),\quad (d,e),$$
$$(d,h),\quad (e,f),\quad (f,g),\quad (g,h)$$

and

$$(1,4),\quad (1,6),\quad (1,7),\quad (1,8),\quad (2,5),\quad (2,7),\quad (3,4),\quad (3,7),\quad (3,8),$$
$$(4,5),\quad (4,6),\quad (4,7),\quad (5,8)$$

Can these two data sheets describe the same department? If not, why not? If yes, which person in the first list corresponds to which person in the second list?

Modeling (ONLY FOR ODD-NUMBERED GROUPS)

Gurdrunsborg is a small community of seven farms. Between each pair of farms is a road. Model the situation using a graph, draw a figure, and explain why there must be a total of $\binom{7}{2} = 21$ roads. These roads only meet at the farms (at all other intersections one road passes above the other on a bridge). Gudrun wants to drive the snowplow so that all roads are plowed, but without ever having to drive the plow along a road that she already plowed. Find a way for Gudrun to do this.

Gustavsholm is an even smaller community with only six farms, again with a road between each pair of farms and no meetings of roads except at te farms. Model the situation using a graph, draw a figure, and explain how many roads there must be in Gustavsholm. Gustav wants to drive the snowplow in the same way as Gudrun, that is, so that all roads are plowed but without ever having to drive the plow along a road that he already plowed . Explain why Gustav will not be able to achieve this, and determine how many roads he can plow at most before driving along a road that he has already plowed.

Reasoning (ONLY FOR EVEN-NUMBERED GROUPS)

Solve exercise 6.8 from the textbook, that is, prove the handshake lemma by induction. Write the proof very carefully.

Calculation (ONLY FOR EVEN-NUMBERED GROUPS)

Construct a binary search tree containing the numbers by Write the numbers from 1 through 20 on pieces of paper and mix them up. By drawing one piece of paper at a time you obtain a random order of these numbers. Write down this order of the numbers. Then construct a binary search tree from this order. Then traverse the tree using in-order, pre-order and post-order, as well as breadth-first and depth-first, writing down the order in which the nodes are visited in each traversal.

Ch. 7

Problem solving (ONLY FOR EVEN-NUMBERED GROUPS)

> *A logic problem: What question should the princess ask?*
>
> A princess visits an island inhabited by two tribes. Members of one tribe always tell the truth, and members of the other tribe always lie.They only answers questions by ”yes” or ”no”.
>
> The princess comes to a fork in the road. She needs to know which way (left or right) leads to the castle where the prince is held captive. She knows that the other way leads to a fire-breathing dragon but she do not know which way is which.
>
> Standing at this fork in the road is a member of each tribe, but the princess can't tell which tribe each of them belongs to. She picks one of the tribe members at random and asks a question. What question should she ask in order to find out which is the way to the castle and how should she interpret the answer she gets?
>
> Note that there are four possible cases: The way to the castle is either to the left or to the right, and the person she asks is either the truth-teller or the liar. Your solution should show, using a table of truth values, that the princess can always deduce from the answer which is the way to the castle.

Modelling (ONLY FOR EVEN-NUMBERED GROUPS)

Imagine that you have acquired a program for logical computations. It can analyze and construct proofs using predicate-logical expressions. You decide to use your program to prove all the theorems in the chapter about graph theory (both those in the text and those in the exercises). In order to do this you want to start by defining predicates that can be useful in this context, and express the theorems using these predicates.

Define a number of predicates that can be relevant in this context (try to make both monadic, dyadic and triadic[2] ones) and express some of the theorems in the chapter using these predicates.

Arguing (ONLY FOR ODD-NUMBERED GROUPS)

Let$P(x, y)$ stand for "x tickles y". Consider the following predicate-logical expressions:

$$(\forall x \forall y P(x, y)) \rightarrow (\forall x P(x, x))$$

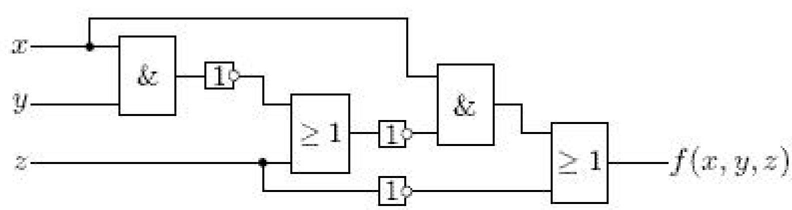
and

$$(\forall x \forall y P(x, y)) \rightarrow (\forall x \forall y P(y, x))$$

In plain English, what do these expressions state? Which of these statements is a tautology, that is, necessarily true? Which of these statements is not necessarily true? Explain!

Computational skills (ONLY FOR ODD-NUMBERED GROUPS)

Here is a logical circuit.



Write down the function computed by this circuit. Find the table of values, and write the function in both conjunctive and disjunctive normal form. Furthermore, make the transformation from the original to the disjunctive normal form using the rules in table 7.3. Draw a circuit that computes the function using the normal form of your choice.

[2]enställiga, tvåställiga och treställiga

Ch. 9

Problem solving (ONLY FOR ODD-NUMBERED GROUPS)

Construct a Mealy machine that delivers coffee. The coffee costs 6 kronor a cup. The machine should accept all Swedish coins (1, 5 and 10 kronor coins) and return any change. If you are in an ambitious mood, add the options of cream and sugar.

Modeling (ONLY FOR ODD-NUMBERED GROUPS)

Specify a Mealy machine that models a system for intelligent traffic lights at a pedestrian crossing of a busy road. There are two kinds of output in transitions between states: Traffic signals can be switched from green to red or vice versa and timers can be set. There are two kinds of input that may trigger transitions between states: Timers can say that the set time has passed and sensors can say that an oncoming car or a waiting pedestrian has been detected. Along with your Mealy machine, describe in plain text how your system works.

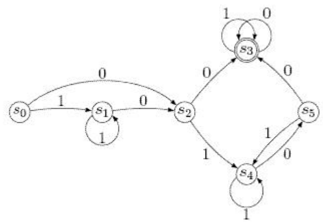
Reasoning (ONLY FOR EVEN-NUMBERED GROUPS)

A nondeterministic finite automaton is a finite automaton in which the same input from the same state may lead to several different possible transitions. A nondeterministic finite automaton accepts an input string if at least one of possible sequence of transitions leads to an accepting state. Prove that for any regular language it is possible to construct a nondeterministic finite automaton with a single accepting state that recognizes the langauge. Hint! Start by showing how to construct an automaton that recognizes a single symbol or λ. Then show how to construct three nondeterministic automata, each with a single accepting state, to recognize the composite languages $R_1 + R_2$, R_1R_2 and R_1^*, given that you already have such automata for the regular languages R_1 och R_2. Explain how an induction argument completes the proof.)

Calculation (ONLY FOR EVEN-NUMBERED GROUPS)

The following automaton recognizes a certain regular language.

From the automaton, derive a regular expression for this language. Then simplify this expression. From the simplified expression, construct a simpler



automaton that recognizes the same language. Describe this language in plain English.

9 Appendix 2: I-report

The I-report is an individual assignment in discrete mathematics. You choose freely an I-report topic from the list below. During the third week of the course you should tell your class teacher what topic you have selected .

Feel free to talk about your assignment with others. However, the report you write should be entirely your own work. Start the report by presenting the problem (for instance, "This report presents a solution to the following problem: ..."). You should not just copy the problem formulation from the list below but rewite it so that it makes sense as an introduction to a report.

Please show a preliminary version of your teachers early, so that you can get feedback and improve your report before submission. A print-out of your report must be submitted to your teacher on the date given in the schedule of the course. If you miss this deadline, or if the report you submit is returned to you for corrections, you can only receive mark 3 on the I-report. The last opportunity to submit a corrected report is one week before the last scheduled lesson. If the report that does not pass by the end of the course, you must retake the course.

You must also email your report to kimmo.eriksson.malard @ analys.urkund.se, which will perform an automatic check for plagiarism against other submitted reports (including previous years) and other sources such as the internet. The examiner will then make an own assessment of whether plagiarism exists.

The I-report must meet the following criteria. Read them carefully!

- Written in a word processor
- Name, personal number and date specified
- No plagiarism
- At most 3 pages in length (excluding figures and appendices)
- A clear description of the problem is included
- A clear description of the solution is included, with acceptable mathematical expressions
- The content is essentially accurate, relevant and complete
- The language is comprehensible, grammatically correct and spelled correctly

I-report topic 1: Colored cards

(mark 3 for task a, mark 5 for tasks a+b)

This task involves decks of cards. Each card is painted with one color on one side and a different color on the other side. All cards have different color combinations.

a. The Little Deck consists of 21 cards with all different color combinations of seven colors (black, white, blue, yellow, green, red, and brown). Show that it is possible to arrange the deck so that the bottom card has the same top color as the bottom color of the next card, which in turn has a top color that is the same as the bottom color of the next card, etc.., all the way up to the top card.

b. The Large Deck consists of the same 21 cards plus an additional 7 cards. The seven additional boards are gray on one side and has the other side painted in one of the seven previously listed colors. Is it still possible to arrange the deck so that the bottom card has the same top color as the bottom color of the next card, which in turn has a top color that is the same as the bottom color of the next card, etc.? If it is possible, you should present such an ordering of the cards. If you think it is not possible, you should present a proof why it's impossible. (Hint: Think of Euler.)

I-report topic 2: The Fibonacci numbers are relatively prime

(mark 5)

The Fibonacci numbers are defined recursively by two initial values, $f_0 = 0$, $f_1 = 1$, and the recursive rule $f_n = f_{n-1} + f_{n-2}$ for all integers $n > 1$. Prove, using the principle of induction and one step of Euclid's algorithm, that $\gcd(f_n, f_{n-1}) = 1$ for all integers $n > 1$.

I-report topic 3: Sum with factorials

(mark 3 for task a, mark 5 for tasks a+b)

a. Calculate the value of

$$s_n = 1 + \sum_{k=1}^{n} k \cdot k!$$

for $n = 1, 2, 3$ and 4. Find a mathematical expression in n that does not include the sum symbol and still gives the correct value of s_n for $n = 1, 2, 3$ and 4.

b. If you chose your expression cleverly, it will give the correct value for all positive integer values of n. Prove by induction that it really does.

I-report topic 4: Sum with binomial numbers

(mark 5)

Consider the sum

$$\sum_{m=k}^{n} \binom{m}{k}$$

Your task is to find a much simpler formula (that does not include the sum symbol) and prove that your formula gives the same value as the sum for all positive integer values of n and k such that $k \leq n$. Hint: Begin by considering the case $k = 3$ and use Pascal's triangle to calculate the sum for $n = 4, 5, 6$. Can you find these values somewhere in Pascal's triangle? This will give you insight in what the expression will look like (namely, as a binomial number involving n and k) and how the formula can be proved (namely, using the relations that hold between numbers in Pascal's triangle).

I-report topic 5: Diophantic equation

(mark 5)

Give a proof by induction (or another kind of strict proof) for the fact that for each integer $n > 43$ there is a nonnegative solution to the Diophantic equation $12x + 5y = n$. By a nonnegative solution is meant that neither x nor y is negative. Hint: Start by finding a nonnegative solution for the first three cases: $n =$ 44,45, 46. Then come up with a way to use these solutions to create solutions for higher values of n.

I-report topic 6: Toss a coin and throw dice

(mark 3 for task a, mark 5 for tasks a+b)

a. A symmetric coin is tossed anumber of times. In every toss there is equal probability for heads and tails. Let p_n denote the probability of having an even number of heads in the first n tosses. Set up a recursion for p_n, i.e., express $p_n$in terms of p_{n-1}. What is p_1? Using the recursion, determine p_2 and then motivate what the value of p_{491}will be.

b. If you throw n standard dice, what is the probability to obtain an even number of sixes?

I-report topic 7: Composition

(mark 3 for task a, mark 5 for tasks a+b)

To write a positive integer as a sum of one or more positive integers is called a composition. For instance, there are four compositions of the number three: $1+1+1$, $1+2$, $2+1$, 3. Compositions can be represented by diagrams with circles and bars: $\circ|\circ|\circ$ represents $1+1+1$, $\circ\circ|\circ$ represents $2+1$, $\circ|\circ\circ$ represents $1+2$ and $\circ\circ\circ$ represents 3. Every composition corresponds to a certain choice of spaces between circles to be filled with bars.

a. Explain, using this representation, why there are 2^{19} compositions of 20. How many of these compositions have only even-sized terms (like 12+2+6)?

b. State and motivate a general principle for the number of compositions of n in which every term is divisible by k.

I-report topic 8: Squares in a chessboard

(mark 3 for task a, mark 5 for tasks a+b+c)

A standard chessboard has 8×8 squares of size 1×1. These unit squares also form larger-sized squares. For instance, there is a single square of size 8×8, and four squares of size 7×7.

a. What is the total number of squares of all sizes that you can find in a chessboard of size $n \times n$? The answer should be well explained and expressed as a sum using the sum symbol.

b. Express the sum as a third degree polynomial. Hint: Calculate the answer for $n = 1, 2, 3, 4$ and then determine the four coefficients of the polynomial $an^3 + bn^2 + cn + d$ by solving the system of four equations you thereby obtain for the coefficients.

c. Use induction to prove that the polynomial and the sum has the same value for all integers $n \geq 1$.

I-report topic 9: Patterns in sums

(mark 3 for task a, mark 5 for tasks a+b)

Consider the following identities:

$1 = 1$

$2 + 3 + 4 = 1 + 8$

$5 + 6 + 7 + 8 + 9 = 8 + 27$

$10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64$

a. Do you see a pattern? State the pattern as a general formula for identity number n. The formula should be on the form $\sum_{i=f(n)}^{g(n)} i = a(n) + b(n)$ so the task is to find expressions for $f(n)$,$g(n)$,$a(n)$ and $b(n)$, and substitute these expressions into this formula.

b. Prove that the formula you stated is true for all integers $n \geq 1$.

I-report topic 10: Nontrivial solutions to equations in $\mathbf{Z}_{60}$

(mark 3 for task a, mark 5 for tasks a+b)

a. Is it possible to assign values to the constant coefficients a, b, c such that for all x i $\mathbf{Z}_{60}$the equation $ax^2 + bx + c = 0$ holds in $\mathbf{Z}_{60}$? Well, clearly one solution is to set $a = 0, b = 0, c = 0$. The real question is whether there is any other solution? Find such a solution or prove that there isn't one. Hint: The case $x = 0$ gives one equation that must be satisfied. The case $x = 1$ gives another equation, etc. .

b. Find a positive integer $n \neq 60$ such that the same question for $\mathbf{Z}_n$has the opposite answer to the question for $\mathbf{Z}_{60}$.

10 Appendix 3: Oral exam

The week after the end of the course offers several opportunities to take the oral exam. See the course schedule for the time and place of each opportunity. There are a limited number of spaces at every opportunity. Sign up for the oral exam using Blackboard, under ”Grupper/Groups”.

Remember to bring your ID to the oral exam.

At the oral exam you will receive a sheet of paper with

- five of the core concepts of the course (taken from the bullet point list of highlights in the beginning of every chapter), and
- five exercises, which will be selected from the belw list of seventy exercises.

With nothing but pencil and blank sheets of paper at your disposal, your task is to explain the concepts and solve the exercises. When you are finished you tell the teacher that you are ready to be examined.

To pass the oral exam with mark 3 you must be able to explain at least three of the five concepts and correctly solve at least three of the five exercises (and respond to questions on your solution such that the teacher can see that you know what you are doing). In case of any errors or misunderstandings the teacher can give you one, but only one, attempt at improving your answers.

To pass the oral exam with mark 5 you must give at least nine correct answers in total.

Exercises will be selected from the following list of 70 exercises.

- 2.6, 2.12, 2.17, 2.19, 2.32, 2.37, 2.38, 2.41, 2.42, 2.59
- 3.9, 3.10, 3.19, 3.20, 3.26, 3.31, 3.36, 3.46, 3.63, 3.68
- 4.2, 4.6, 4.7, 4.11, 4.12, 4.18, 4.20, 4.21, 4.23, 4.47
- 5.3, 5.13, 5.20, 5.32, 5.34, 5.35, 5.39, 5.43, 5.57, 5.83
- 6.4, 6.13, 6.26, 6.33, 6.48, 6.49, 6.61, 6.73, 6.82, 6.86
- 7.5, 7.9, 7.16, 7.24 ,7.33, 7.37, 7.40, 7.47, 7.53, 7.57,
- 9.3, 9.9, 9.11, 9.13, 9.16, 9.23, 9.25, 9.26, 9.32, 9.45