1

Set $f(x) = \tan(x^2)$ and $g(x) = \sin(x)$. We see that $f(0) = g(0) = 0$. Thus, direct substitution does not work but our functions are such that we can try l'Hopital's rule.

We get

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2x \cdot (1 + \tan^2(x^2))}{\cos(x)} = \frac{0 \cdot 1}{1} = 0.$$

Thus $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.

Answer: 0

2

We know that f has an absolute maximum since it is a continuous function on a closed interval. We also know that it attains this maximum value at one of the endpoints or at a stationary point. So we solve for the stationary points.

We find $f'(x) = \frac{1-x^2}{(1+x^2)^2}$. We determine for what x in $(0, 10)$, $f'(x)$ is zero.

$$f'(x) = 0 \Leftrightarrow 0 = 1 - x^2 \Leftrightarrow x^2 = 1 \Leftrightarrow x = 1.$$

$x = -1$ is not possible since $x \in (0, 10)$.

We now check the value of f in our stationary point, and at the endpoints.

$$f(0) = 0$$
$$f(10) = \frac{10}{101}$$
$$f(1) = \frac{1}{2}.$$

Answer: The absolute maximum is $\frac{1}{2}$.

3

We use integration by parts.

$$\int_0^\pi x\cos(x)\,\mathrm{d}x = [x\sin(x)]_0^\pi - \int_0^\pi \sin(x)\,\mathrm{d}x = 0 - [-\cos(x)]_0^\pi = -1 - 1 = -2.$$

Answer: -2.

4

An integrating factor is e^x. We multiply the equation by this factor and rewrite.

$$e^x y' + e^x y = 1 \Leftrightarrow (e^x y)' = 1 \Leftrightarrow e^x y = x + C \Leftrightarrow y = xe^{-x} + Ce^{-x}.$$

Answer: $y = xe^{-x} + Ce^{-x}$, where C is an arbitrary constant.

5

We apply the ratio test. Set $a_n = n^{1/2} x^n$. Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot |x| = \lim_{n\to\infty} \sqrt{1 + \frac{1}{n}} \cdot |x| = 1 \cdot |x| = |x|.$$

By the ratio test the series converges if $|x| < 1$ and diverges if $|x| > 1$.

Answer: The radius of convergence is 1.