1

Set $f(x) = \cos(2x) - \cos(x)$ and $g(x) = \sin(x)$. We see that $f(0) = g(0) = 0$. Thus, direct substitution does not work but our functions are such that we can try l'Hopital's rule.

We get

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{-2\sin(2x) + \sin(x)}{\cos(x)} = \frac{0+0}{1} = 0.$$

Thus $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$.

Answer: 0

2

We know that f has an absolute maximum since it is a continuous function on a closed interval. We also know that it attains this maximum value at either one of the endpoints or at a stationary point. So we solve for the stationary points.

We find $f'(x) = 1 - 3x^2$. We determine for what x in $(0, 10)$, $f'(x)$ is zero.

$$f'(x) = 0 \Leftrightarrow 1 = 3x^2 \Leftrightarrow x^2 = \frac{1}{3} \Leftrightarrow x = \frac{1}{\sqrt{3}}$$

$x = \frac{-1}{\sqrt{3}}$ is not possible since $x \in (0, 10)$.

We now check the value of f in our stationary point, and at the endpoints.

$$f(0) = 0$$
$$f(10) = -990$$
$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$$

Answer: The absolute maximum is $\frac{2}{3\sqrt{3}}$.

3

We begin by simplifying the numerator of the integrand.

$$\cos(2x) + 2\sin^2(x) = \cos^2(x) - \sin^2(x) + 2\sin^2(x) = \cos^2(x) + \sin^2(x) = 1.$$

Thus our integral becomes

$$\int_0^2 \frac{\cos(2x) + 2\sin^2(x)}{e^x} \, \mathrm{d}x = \int_0^2 \frac{1}{e^x} \, \mathrm{d}x = \int_0^2 e^{-x} \, \mathrm{d}x = -e^{-x}\Big]_0^2 = 1 - e^{-2}.$$

Answer: $1 - e^{-2}$

4

We rewrite our differential equation into linear form.

$$e^{-x}y' + 2y = 0 \Leftrightarrow y' + 2e^{x}y = 0.$$

We see that we can apply the method of integrating factor. A primitive function to $2e^x$ is $2e^x$ so e^{2e^x} is an integrating factor. We find that

$$y' + 2e^{x} = 0 \Leftrightarrow e^{2e^x}y' + 2e^{x}e^{2e^x}y = 0 \Leftrightarrow \left(ye^{2e^x}\right)' = 0$$

By taking primitive functions on both sides we get the following

$$\left(ye^{2e^x}\right)' = 0 \Leftrightarrow ye^{2e^x} = C \Leftrightarrow y = Ce^{-2e^x},$$

where C is an arbitrary constant.

Answer: $y = Ce^{-2e^x}$, where C is an arbitrary constant.

5

We apply the root test. We find that (with $a_n = \frac{e^{2n+1}}{n^n}$)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{e^{2+1/n}}{n} = 0.$$

The rightmost equality above holds since $e^{2+1/n}$ is bounded. The root test thus shows that the series is convergent.

Answer: The series is convergent.