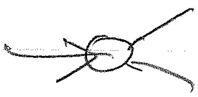
Algorithmic problems in graph theory

Definition. A cycle C in a graph G is said to be *Hamiltonian* if its length is $|V(G)|$ (i.e., if $V(C) = V(G)$).
A graph is said to be Hamiltonian if it has some Hamiltonian cycle.

Theorem: It is NP-complete to decide if a general graph is Hamiltonian. It is equally NP-complete to decide if a 3-regular graph is Hamiltonian.

Basic problem: Has a choice on how to proceed from a vertex, but may face consequences only much later.



Let a graph G and a function $L: E(G) \to \mathbb{R}$ be given. $L(e)$ is called the "length" of the edge e.
The *length of a walk* $v_0 e_1 v_1 \ldots e_n v_n$ is $\sum_{k=1}^{n} L(e_k)$.

Shortest path problem:
Given vertices u and v, find a shortest path from u to v.
Solvable in $O(n^2)$ time using Dijkstra's algorithm.

Application: Find cheapest journey from A to B.
Graph vertices are pairs (location, time). Graph is directed.
Edges are connections, length is price of a ticket for that connection. Also edges (l, t_1) to (l, t_2) if $t_1 < t_2$ with length 0 (waiting is free).

Travelling Salesman Problem (TSP):

Given $L: E(K_n) \to \mathbb{R}$, find shortest Hamiltonian cycle.

This is NP-hard, because for any n-vertex subgraph G of K_n one can define
$$L(e) = \begin{cases} 1 & \text{if } e \in E(G) \\ 2 & \text{if } e \notin E(G). \end{cases}$$
There is a L-length n Hamiltonian cycle in K_n iff G has a Hamiltonian cycle.

Best ~~known~~ polynomial approximation algorithm finds a cycle no longer than $\frac{4}{3}$ ~~of~~ the optimal cycle.

Colouring

Let a graph G be given. A function $f: V(G) \to \{1, 2, \ldots, k\}$ is said to be a (proper) k-(vertex-) colouring of G if for any $uv \in E(G)$ it holds that $f(u) \neq f(v)$.
A graph that has a proper k-colouring is said to be k-colourable.
[Example

To test whether a general graph G is 2-colourable can be done in $O(|E(G)|)$ time.
———— // ———— 3-colourable is NP-complete!