Spectral graph theory is the study of graphs by using properties of the adjacency matrix, particularly the eigenvalues.

The following holds for undirected multigraphs. Let $n = |V(G)|$

Observation 1: $A(G)$ is symmetric, so all its eigenvalues are real. Moreover, it has a set of n orthogonal eigenvectors.

Theorem: $\delta(G) \leq \frac{2|E(G)|}{|V(G)|} \leq \lambda_{max}(G) \leq \Delta(G)$.

Proof: Let λ be any eigenvalue of $A(G)$ and x be a corresponding eigenvector. Let j be such that $x_j = \max_i x_i$. Then

$$\lambda x_j = (Ax)_j = \sum_{v_i \in N(v_j)} x_i \leq d(v_j) x_j \leq \Delta(G) x_j \qquad (*)$$

and so in particular $\lambda_{max} \leq \Delta(G)$.

By linear algebra, λ_{max} is the maximal value attained by $x^T A(G) x$ for vectors x of length 1. Hence

$$\lambda_{max} \geq \frac{\mathbb{1}^T}{\sqrt{n}} A(G) \frac{\mathbb{1}}{\sqrt{n}} = \frac{1}{n} \mathbb{1}^T A(G) \mathbb{1} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} A(G)_{ij} =$$

$$= \frac{1}{n} \cdot 2|E(G)|.$$

QED.

Corollary: $\lambda_{max}(G) = \Delta(G)$ if and only if G has a $\Delta(G)$-regular component. The multiplicity of $\Delta(G)$ as eigenvalue is equal to the number of $\Delta(G)$-regular components.

Proof. $\lambda_{max}(G) = \Delta(G)$ requires equality in $(*)$ above, so one must have $d(v_j) = \Delta(G)$ and $x_i = x_j$ for all $v_i \in N(v_j)$. This implies $d(v_i) = \Delta(G)$ for all v_i in the same component as v_j. One eigenvector has $x_i = 1$ for all v_i in that component and $x_i = 0$ outside it.

An important problem in large-scale networks is efficient dissemination of information; you don't want bottlenecks where most who know are only friends with others who already knows.

<u>Definition</u>: An <u>(n,k,c)-magnifier</u> is an n-vertex graph G with $\Delta(G) \leq k$ and such that for every $S \subseteq V(G)$ with $|S| \leq \frac{n}{2}$ it holds that $|N(S) \setminus S| \geq c \cdot |S|$.

Note: $c \leq 1$, because $c \leq \frac{|V(G) \setminus S|}{|S|}$.

But <u>constructing</u> a family of such graphs with fixed k, growing n and c bounded from below is nontrivial. On the other hand, it turns out they are pretty common, so if only one could efficiently test graphs for being magnifiers, then one could find quite a lot of them.

<u>Theorem</u>. Let G be a connected k-regular n-vertex graph. Let λ be the second largest eigenvalue of $A(G)$. For any $S \subseteq V(G)$, denote

$$E(S,\bar{S}) = \{e \in E(G) \mid e = uv \text{ where } u \in S \text{ and } v \notin S\}.$$

Then $$|E(S,\bar{S})| \geq (k-\lambda)\,|S|\,(n-|S|) \cdot \frac{1}{n}$$

Proof. For any vector $x \in \mathbb{R}^n$,

$$x^T(kI-A)x = k\sum_{i=1}^{n} x_i^2 - 2\sum_{ij\in E(G)} x_i x_j = \sum_{ij\in E(G)} (x_i^2 + x_j^2 - 2x_i x_j) =$$
$$= \sum_{ij\in E(G)} (x_i - x_j)^2$$

Now let $s = |S|$. Let $x_i = -(n-s)$ for $i \in S$ and $x_i = s$ for $i \notin S$. Then

$$\sum_{ij\in E(G)} (x_i - x_j)^2 = \sum_{ij\in E(G)} \begin{cases} n^2 & \text{if } ij \in E(S,\bar{S}) \\ 0 & \text{otherwise} \end{cases} = n^2 |E(S,\bar{S})|.$$

Furthermore $\sum_{i=1}^{n} x_i = s\cdot(n-s) - (n-s)\cdot s = 0$, and so x is orthogonal to $\mathbb{1}$, which is the eigenvector of $A(G)$ with eigenvalue k. Hence

$$x^T A(G) x \leq \lambda\, x^T x,$$

and thus
$$n^2|E(S,\bar{S})| = x^T(kI - A(G))x = k x^T x - x^T A(G) x \geq$$
$$\geq (k-\lambda)\, x^T x.$$

Furthermore
$$x^T x = \sum_{i=1}^{n} x_i^2 = s\cdot(-(n-s))^2 + (n-s)\cdot s^2 =$$
$$= s(n-s)((n-s)+s) = s(n-s)n.$$

Therefore
$$|E(S,\bar{S})| \geq (k-\lambda)\, s(n-s)\frac{n}{n^2} = (k-\lambda)|S|(n-|S|)\frac{1}{n}$$

QED

Corollary: A graph such as in the theorem is an (n, k, c)-magnifier, for $c = (k-\lambda)/2k$.

Proof: If $S \subseteq V(G)$ has $|S| = s \leq \frac{n}{2}$, then there are at least $(k-\lambda)s(n-s)/n$ edges from S to $V(G)\setminus S$. No vertex of $V(G)\setminus S$ can receive more than k of these, so
$$|N(S)\setminus S| \geq \frac{(k-\lambda)\,s(n-s)}{k\cdot n} = 2c\cdot s\cdot \frac{(n-s)}{n} \geq c\cdot|S|.$$
QED

Definition. An (n, k, c)-expander is a $\overbrace{\text{graph}}^{\text{(bipartite)}}$ G with $n+n$ vertices, $\Delta(G) \leq k$, and the property that

$V(G) = X \cup Y$
$$|N(S)| \geq \left(1 + c\left(1 - \frac{|S|}{n}\right)\right)\cdot|S|$$

for every $S \subseteq X$ with $|S| \leq \frac{n}{2}$.

Construction: Let G be a (n, k, c)-magnifier with $V(G) = \{v_1, \dots, v_n\}$. Let H be the bipartite graph with $V(H) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ such that
$$x_i y_j \in E(H) \Leftrightarrow v_i v_j \in E(G) \text{ or } i = j.$$
Then H is a $(n, k+1, c)$-expander.

Definition. An n-superconcentrator is an acyclic digraph with n sources and n sinks, such that there for every set A of sources and every set B of $|A|$ sinks has $|A|$ disjoint paths from A to B.