Seminar assignment 1

Each seminar assignment consists of two parts, one part consisting of $3-5$ elementary problems for a maximum of 10 points from each assignment. For the second part consisting of problems in applications you are to choose one of four problems to solve. This part can give up to 5 points from each assignment. Each seminar assignment can give a maximum of 15 points, to pass you will need at least 20 points total from both the assignments.

The first part consists of elementary questions to make sure that you have understood the basic material of the course while the second part consists of one larger application example.

Solutions can either be handed in by email to *christopher.engstrom@mdh.se* or you can hand in handwritten solutions in the box outside Christophers room $U3:185$ before 10.00 Monday 3rd February

1 Part 1

In the first part you are to solve and hand in solutions to the questions. You are allowed to use computer software to check your results, but your calculations as well as your result should be included in the answers for full points.

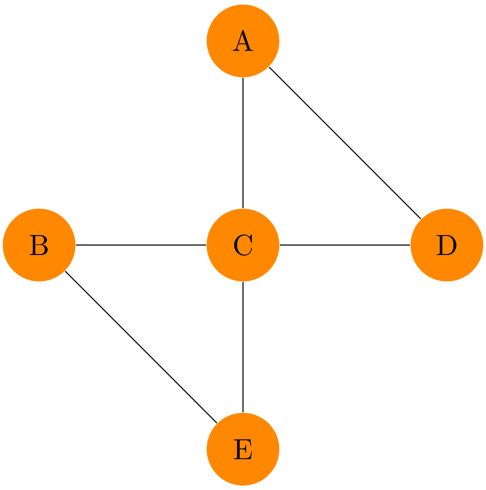
1.1

- (1) Use the characteristic polynomial $\det(\mathsf{I}\lambda - \mathsf{A})$ to find the eigenvalues of A.
- (1) Calculate the Determinant of A

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ 2 & -1 & 0 \end{bmatrix}$$

1.2

- (1) Define the Laplacian matrix for a undirected graph.
- (1) Calculate the Laplacian matrix for the graph below.



1.3

Consider

$$\mathsf{A} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 1/4 & 0 & 0 & 3/4 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- (1) Draw the directed graph described by A.
- (1) Consider the Markov chain described by A. If we start in state 1, what is the probability that we are in state 3 after both 3 and 6 steps?
- (1) What is the probability that we are in state 3 after 6 steps, but where not in state 3 after 3 steps?

1.4

After Calculating the (left) eigenvalues and eigenvectors of A in the previous question we got the following vectors as columns of V.

$$\mathsf{V} = \begin{bmatrix} -0.51 & 0.64 & 0.64 & 0.62 \\ 0.65 & -0.15 - 0.37i & -0.15 + 0.37i & 0.31 \\ -0.46 & -0.39 - 0.13i & -0.39 + 0.13i & 0.52 \\ 0.32 & -0.10 + 0.50i & -0.10 - 0.50i & 0.49 \end{bmatrix}$$

And corresponding eigenvalues in

$$\{-0.39, \ -0.30 + 0.74i, \ -0.30 - 0.74i, \ 1\}$$

(both rounded to 2 decimals):

- (1) Use Perron-Frobenius theorem to find the Perron-Frobenius eigenvalue and corresponding eigenvector. Motivate your answer.
- (1) Give the definition of a Primitive matrix, use this to show that A is primitive.

1.5

We consider the matrices $\mathsf{A}, \mathsf{L}, \mathsf{U}$ with LU-factorization $\mathsf{A} = \mathsf{LU}$.

$$\mathsf{A} = \begin{bmatrix} 3 & 0 & 1 \\ -3 & -4 & 1 \\ 6 & -8 & 15 \end{bmatrix}, \ \mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -2 & 0 \\ 2 & -4 & 3 \end{bmatrix}, \ \mathsf{U} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

- (1) Use the LU-factorization given above to solve the system $\mathsf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [-1\ 1\ 2]^\top$.

2 Part 2

In this second part you are to choose **ONE** example where you attempt to solve the questions presented. If you hand in answers to more than one choice you will get points corresponding to the choice which would give the least total points.

You are allowed to use computer software (and depending on which option you choose might be needed for some of the questions). If you are using a computer for some calculations you should set up the problem and present how it could theoretically be solved by hand. For example you could write "I solved the linear system $Ax = b$ using Matlabs "fsolve", another method would have been to use Gaussian elimination and solving the resulting triangular system".

2.1 SIS-model of an epidemic

Note: due to the fairly large size of matrices in this example you are likely needed to use computer aid when doing some of your calculations.

SIR stands for susceptible-infected-susceptible, representing a model with constant population size where someone getting infected and then later recovering does not become immune to the disease (hence can get infected again). There is no period between getting infected and becomming infectious, instead any infected individual are assumed to also be infectious and can pass on the disease.

We model the possible epidemic using a Markov chain where every step is considered short enough that at most one individual can be infected or recover. Each state I_i in the Markov chain represent i infected people out of a total polulation of N people.

Assuming every infected individual on average makes enough contakt to infect β people per time step. The probability for each of these people to be susceptible is $(N - i)/N$. Multiply this with the number of infected individuals i and we have a probability $\beta i(N -$

$i)/N$ someone new is infected. We assume the probability for someone to recover to be proportional to the number of infected people αi. This gives:

$$P(X_{n+1} = i + 1|X_n = i) = \beta i(N - i)/N$$
$$P(X_{n+1} = i - 1|X_n = i) = \alpha i$$
$$P(X_{n+1} = i|X_n = i) = 1 - \beta i(N - i)/N - \alpha i$$

- (a) For $N = 5$ write out the transition matrix for this Markov chain.

Eventually the epidemic will go extinct, we want to know how long we can expect it to take. We remember we could find the hitting time k_i^A as the minimal solution to:

$$\begin{cases} k_i^A = 0, & i \in A \\ k_i^A = 1 + \sum_{j=1}^{n} p_{i,j} k_j^A & i \notin A \end{cases}$$

- (b) Using $N = 20, \alpha = 0.012, \beta = 0.01$ Find the expected time until a the epidemic dies out if we start with 1 infected individual (k_1^0).
- (c) Calculate the hitting probabilities to find the probability that the epidemic gets so large large that at one point there is at least 10 individuals infected at the same time.
  Hint: You don't need to consider the states corresponding to more than 10 infected people.

In order to calculate stationary distributions we make one additional assumption, we assume that when there is only 1 infected individual, that individual do not recover (hence we can never end up with no infected individuals).

- (d) Change your system as discussed above and calculate the stationary distribution. Interpret the results.
- (e) What is the expectation for the number of infected people after a long time?

2.2 Modelling of student numbers

Our aim is to create a model that describes the flow of students through some 3 year program at a University. After following all the students on the program over one 1 year we could compile the following data of years they had studied at the start of the year, and if they continued their studies in the same program, graduated or quit their studies during the year.

starting year of studies	Start	Continued studies next year	Graduated	quit
< 1	100	70	0	30
1-2	50	40	0	10
2-3	50	15	30	5
3+	20	2	16	2

Given a table as above we can then estimate the transition probabilities of a Markov chain describing the movement among different groups as:

$$\mathsf{P}_{i,j} = \frac{\mathsf{G}_{i,j}}{\mathsf{S}_i}$$

Where $\mathsf{G}_{i,j}$ is the number of students that ended up in group j starting in group i and S_i is the number of students starting in i at the start of the interval.

- (a) Using the table above, esimate the transition probabilities for a Markov chain with 6 states (students on their 1st,2nd,3rd,or later year of study as well as two absorbing states Graduated and Quit studies) which describes a single student and create a stchastic matrix for our model.
- (b) What is the probability that a student starting on their first year of studies eventually graduates?
  Hint: look up how to calculate the hitting probabilities.
- (c) How many years on average does a student study on the program before they either graduate or quit? Assume they only quit/graduate at the end of an interval.

Next we want to make a change in the model in order to estimate the number of students in the different year categories after some time. To do this we assume there is a constant number of student studying on the program, whenever a student graduates or quit, a new student takes their place.

- (d) Change your Markov chain such that the "Graduate" and "Quit" states are no longer absorbing but instead leads to a new student starting in their first year. Calculate the stationary distribution of this Markov chain.
- (e) Interpret the results, how many students would need to start every year if we want on average 50 students to graduate every year.

2.3 Fail-Safe system

Consider a machine with two independent controls A, B designed to immediately stop the machine if something bad is about to happen. Whenever the machine is stopped by A or B the problem can quickly be fixed and the machine restarted almost immediately. However if both controls fails to stop the machine at the same time something bad have happened to either the machine or the operator. We consider the discrete time points $t_1, t_2, \ldots$ where the machine needs to be stopped.

If one control fails to activate it is immediately replaced with a new one before the next time point. A control that worked during the last time of activation t_n have a 95% chance to work at the next time it needs to activate t_{n+1}. However immediately after a control fails, the new untested replacement have only a 80% to activate at the next time step, if it works correctly in that step it's considered "tested" and works with 95% reliability after that.

We describe this as an absorbing Markov chain with 4 states $\{(1,1),(1,0),(0,1),(0,0)\}$ where $(1,1)$ correspons to both controls working, $(1,0),(0,1)$ corresponds to the first or last control working and $(0,0)$ corresponds to the case with both controls failing at the same time.

- (a) Order the states as above and let $(0,0)$ correspond to an absorbing state, construct the transition matrix P for this Markov chain.

Since $(0,0)$ is absorbing and this state can be reached from any other state, the machine will obviously eventually fail. The obvious question is how long can we expect it to take? As we ordered the states P should already be in canonical form.

- (b) What is the expected number of steps until the machine fails if we start with two tested controls?
  Hint: look up how to calculate Hitting times.

Next we consider a slightly different problem. Now the times which the controls needs to be activated occur between regular intervals of the same length. We consider the discrete time points $t_1, t_2, \ldots$ as the points i time between intervals. If the machine fails at one point it is stopped during the whole next interval during which it is repaired and properly tested such that in can be considered to be in in the "tested" state after.

To accomodate this change we change our transition matrix such that we always move to $(1,1)$ after we get to $(0,0)$. If we have the states in the same order as earlier we can do this simply by changing the last row to [1 0 0 0].

Now this new transition matrix is irreducible and primitive and we can use for example Perron-Frobenius to find it's stationary distribution.

- (c) Find the stationary distribution of our new Markov chain, What is the probability of the machine being in the down state $(0,0)$ after a long time?

We assign a reward $r > 0$ whenever the machine is running (all except state $(0,0)$) and a cost $c \geq 0$ to repair a failed 1 control regardless whether the machine is running or not. We then get total reward R in one step as:

$$R(X_n) = \begin{cases} r, & X_n = (1,1) \\ r - c, & X_n \in \{(1,0),(0,1)\} \\ -2c, & X_n = (0,0) \end{cases}$$

We can then find the expected reward per step after a long time as $E(X_n) = \sum_{i=1}^{4} R(\pi_i)$ where π is the stationary distribution.

- (d) Find the long term expected reward of one step in the 2 control system.
- (e) Construct a similar reward for the 1 control system and calculate the long term expected reward for that in the same way. At what values $c, r > 0$ would it be more profitable to use the 2 control system over the 1 control system?