Seminar assignment 1

Each seminar assignment consists of two parts, one part consisting of $3-5$ elementary problems for a maximum of 10 points from each assignment. For the second part consisting of problems in applications you are to choose one of four problems to solve. This part can give up to 5 points from each assignment. Each seminar assignment can give a maximum of 15 points, to pass you will need at least 20 points total from both the assignments.

The first part consists of elementary questions to make sure that you have understood the basic material of the course while the second part consists of one larger application example.

Solutions can either be handed in by email to *christopher.engstrom@mdh.se* or you can hand in handwritten solutions either during the lectures or in the box outside Christophers room $U3:185$.

1 Part 1

In the first part you are to solve and hand in solutions to the questions. You are allowed to use computer software to check your results, but your calculations as well as your result should be included in the answers for full points.

1.1

- (1) Use the characteristic polynomial $\det(\mathsf{I}\lambda - \mathsf{A})$ to find the eigenvalues of A.
- (1) Calculate the trace of A and verify that $\mathrm{tr}(\mathsf{A}) = \sum_{i=1}^{n} \lambda_i$.

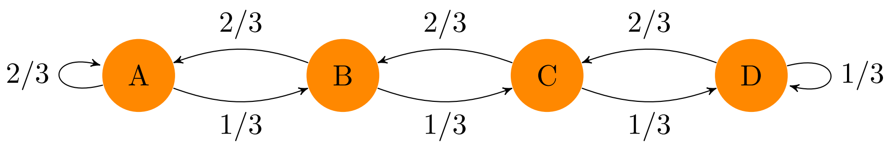
$$\mathsf{A} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

1.2

- (1) Draw the undirected graph described by the adjacency matrix A.
- (1) What is the Laplacian matrix of the graph described by A? Ignore any loops in the graph (vertices with edges to themselves).

$$\mathsf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

1.3



- (1) What is the corresponding stochastic matrix to the Markov chain described by the above graph?
- (1) If we start in state A, what is the probability that we are in state B after 3 steps?
- (1) What do we mean by that Markov chains are "memoryless"? Use this property to find the probability to be in state A after 6 steps if we know that we where in state A after 3 steps.

1.4

- (1) A is a irreducible matrix (you do not have to show this), without calculating the eigenvalues: use the Perron-Frobenius theorem to give an estimate of the value of the largest eigenvalue of A.

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

1.5

We consider the matrices $\mathsf{A}, \mathsf{L}, \mathsf{U}, \mathsf{P}$ with LUP-factorization $\mathsf{PA} = \mathsf{LU}$.

$$\mathsf{A} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 0 & 2 \end{bmatrix}, \ \mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \ \mathsf{U} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}, \ \mathsf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- (1) Give 3 properties of unitary matrices and show that P is a unitary matrix.
- (1) Use the LUP-factorization given above to solve the system $\mathsf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [1\ 0\ 1]^\top$.

2 Part 2

In this second part you are to choose **ONE** example where you attempt to solve the questions presented. If you hand in answers to more than one choice you will get points corresponding to the choice which would give the least total points.

You are allowed to use computer software (and depending on which option you choose might be needed for some of the questions). If you are using a computer for some calculations you should set up the problem and present how it could theoretically be solved by hand. For example you could write "I solved the linear system $Ax = b$ using Matlabs "fsolve", another method would have been to use Gaussian elimination and solving the resulting triangular system".

2.1 Modeling the World Income Distribution

Our aim is to attempt to model the distribution of wealth among countries using historical data and a Markov model.

Countries are divided in 3 groups representing high,medium and low Real gross domestic product (GDP) per worker. Countries are divided in the groups by comparing them to a large influential country (such as the US).

We want to create a transition matrix for the countries in these states by following their movement along the groups over a period of 20 years. Our study involves data on 41 countries where we consider them to have high,medium or low income at the beginning of the interval and then see in what group they end up at the end of the interval.

	S	G_{high}	G_{medium}	G_{low}
high	10	9	1	0
medium	20	2	15	3
low	11	0	1	10

Given a table as above we can then estimate the transition probabilities of a Markov chain describing the movement among different groups as:

$$\mathsf{P}_{i,j} = \frac{\mathsf{G}_{i,j}}{\mathsf{S}_i}$$

Where $\mathsf{G}_{i,j}$ is the number of countries that ended up in group j starting in group i and S_i is the number of countries starting in i at the start of the interval.

- (a) Given the table above, estimate the transition probabilities over a 20 year period and create a stochastic transition matrix for our model.
- (b) Calculate the stationary distribution of this Markov chain and interpret the result.

While it's obvious in our model that any low income country will eventually become a high income country (and the opposite as well). We are interested in seeing how long we can expect a low income country to become a high income country. We remember we could find the hitting time k_i^A as the minimal solution to:

$$\begin{cases} k_i^A = 0, & i \in A \\ k_i^A = 1 + \sum_{j=1}^{n} p_{i,j} k_j^A & i \notin A \end{cases}$$

- (c) Find the expected time until a country starting with a low income becomes a country with a high income, (k_{low}^{high}).

This first model have one weakness in that it models countries among eachother regardless of the population of said countries. In order for the model to reflect the income of people over the period instead we instead look at the proportion of people living in high,medium and low income countries respectively. To help with this we look at the population in each group instead:

	S	G_{high}	G_{medium}	G_{low}
high	200	195	5	0
medium	400	25	350	25
low	200	0	90	110

Where the numbers are in terms of million people and populations are taken at the end of the interval.

- (d) Create the new transition matrix, give at least one (possibly unreasonable) assumption we need to make about the population for this new Markov chain.
- (e) Calculate the new stationary distribution, how does it compare to your previous result?

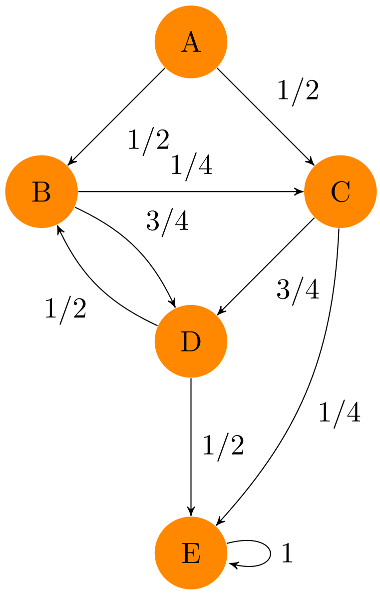
2.2 Program segmentation

As a program grows in size it might happen that the whole program no longer fit in the working memory of the coputer. Then the program need to be partitioned in smaller parts loaded from the slow bulk memory (such as the disc) and executed one at a time. However if we partition in to small segments we risk slowing down the program because of many reads from the bulk memory.

If the program have no cycles we can do a suitable partition fairly easily by sorting the pieces topologically. Starting from the first piece we partition the program in as large partitions as possible to still fit in working memory sorted topologically. As we have no cycles, once we are finished executing one part of the program we know that we never need to execute that same part again.

If there are cycles we need another approach such as using a Markov model as we will try to do here. By assuming the program behaves as a Markov system we can estimate the transition probabilities of going from one part of the program to another (for example by executing the program a number of times using some sample data). We get a graph describing these transition probabilities between parts of the program as below.

Where we set the end-state 'E' as a absorbing state.



- (a) Find the transition matrix for this program, write it in canonical form.

While the stationary distribution is not interesting (we will always end up in the absorbing state at the end), how many times we can expect a certain part of the program is. To find the expected number of times a part of the program is executed we want to sum the probability that we are in this part for every step in the Markov chain. If X is the part of the transition matrix corresponding to the transient states we get $\mathsf{I} + \mathsf{X} + \mathsf{X}^2 + \ldots = (\mathsf{I} - \mathsf{X})^{-1}$ (Neumann series since $\mathsf{X}^n \to \mathsf{0}$, as $n \to \infty$) with elements $(\mathsf{I} - \mathsf{X})^{-1}_{i,j}$ holding the expected number of timeswe execute j starting in i.

- (b) What is the expected number of times we need to execute the code in 'B' in a typical run of the program (starting in 'A')?

- (c) What is the probability that the program ever execute the code in 'D' (starting in 'A')?

To every part of the program we have one other important piece of information, namely the average execution time and memory requirement of said part. You can see this information for our program in the table below.

	A	B	C	D	E
execution time	15s	20s	35s	45s	80s
required memory	200mb	100mb	150mb	200mb	100mb

- (d) Assuming no parallell processing, what is the expected execution time of the whole program?

Lets assume our computer have 400mb available working memory, we want to partition our program in as few partitions as possible, where each partition needs at most 400mb of memory. We also don't want to minimize the chance that we need to load a partition as much as possible (visit ony of the vertices in that partition). One method to do this is by repeatedly delete the edges we are least likely to visit untill we can partition the graph such that the partitions are topologically ordered (once we leave a partition we can never return to the same partition).

- (e) Use your previous answers to find the expected visits of all vertices and delete the edges we are least likely to visit. Partition the graph in 2 partitions both fitting in work memory using the method above. (HINT: you will have to remove 2 edges).

2.3 SIS-model of an epidemic

Note: due to the fairly large size of matrices in this example you are likely needed to use computer aid when doing some of your calculations.

SIR stands for susceptible-infected-susceptible, representing a model with constant population size where someone getting infected and then later recovering does not become immune to the disease (hence can get infected again). There is no period between getting infected and becomming infectious, instead any infected individual are assumed to also be infectious and can pass on the disease.

We model the possible epidemic using a Markov chain where every step is considered short enough that at most one individual can be infected or recover. Each state I_i in the Markov chain represent i infected people out of a total polulation of N people.

Assuming every infected individual on average makes enough contakt to infect β people per time step. The probability for each of these people to be susceptible is $(N-i)/N$. Multiply this with the number of infected individuals i and we have a probability $\beta i(N-i)/N$ someone new is infected. We assume the probability for someone to recover to be proportional to the number of infected people αi. This gives:

$$P(X_{n+1} = i+1 | X_n = i) = \beta i(N-1)/N$$
$$P(X_{n+1} = i-1 | X_n = i) = \alpha i$$
$$P(X_{n+1} = i | X_n = i) = 1 - \beta i(N-1)/N - \alpha i$$

- (a) For $N = 5$ write out the transition matrix for this Markov chain.

Eventually the epidemic will go extinct, we want to know how long we can expect it to take. We remember we could find the hitting time k_i^A as the minimal solution to:

$$\begin{cases} k_i^A = 0, & i \in A \\ k_i^A = 1 + \sum_{j=1}^{n} p_{i,j} k_j^A & i \notin A \end{cases}$$

- (b) Using $N = 20, \alpha = 0.008, \beta = 0.01$ Find the expected time until a the epidemic dies out if we start with 1 infected individual (k_2^0).
- (c) Calculate the hitting probabilities to find the probability that the epidemic gets so large large that at one point there is at least 10 individuals infected at the same time. Hint: You don't need to consider the states corresponding to more than 10 infected people.

In order to calculate stationary distributions we make one additional assumption, we assume that when there is only 1 infected individual, that individual do not recover (hence we can never end up with no infected individuals).

- (d) Change your system as discussed above and calculate the stationary distribution. Interpret the results.
- (e) What is the expectation for the number of infected people after a long time?

2.4 Modelling power output of a wind turbine

Note: due to the fairly large size of matrices in the second half of this example you are likely needed to use computer aid when doing some of your calculations.
Our aim is to create a (simple) Markov model for the generated power of a wind turbine. However we start by looking at the wind speeds and create a Markov model for that. We measure the wind speed every 30min. for about 3 days (a total of 145 measurements) and put the measured speed in one of four groups (low,medium,high,storm) depending on measured wind speed $(0-2m/s), (2-10m/s), (10-30m/s), (30+m/s)$. Rather than looking at the measures directly we instead looked at the change in wind speed between any two consecutive measures and compiled the result in the table below.

	low	medium	high	storm
low	15	5	0	0
medium	5	30	10	0
high	0	10	40	5
storm	0	0	5	20

- (a) Create a stochastic matrix for this Markov chain model with transition probabilities $T_{i,j} = r_{ij}/r_i$ where $r_{i,j}$ is the number of observed transitions from state i to state j and r_i is the total number of observed transitions from state i.
- (b) Find the stationary distribution of this Markov chain.

The power generated can be modelled by:

$$P(v) = \begin{cases} 0 & v < v_{min} \\ Cv^3 & v_{min} \leq v \leq v_{opt} \\ Cv_{opt}^3 & v_{opt} < v \leq v_{max} \\ 0 & v_{max} < v \end{cases}$$

Where for our wind turbine $v_{min} = 2m/s$ is the minimum wind speed required to generate power, $v_{opt} = 10m/s$ is the wind speed needed for the wind turbine to be working at maximum performance and $v_{max} = 30m/s$ is the maximum wind speed for which the wind turbine can run.

- (c) When in "medium" assume an average wind speed of $(v_{min} + v_{opt})/2$, use this to calculate the expectation of the generated power over a single day.

  (Expectation $E = \sum_{i=1}^{4} p_i x_i$ where $p_i$is the probability to be in state i and x_i is the power generated in state i).

While the expected power output might be good, the Turbine is generating no power a significant portion of the time. In order to try and get a more reliable power output a second wind turbine is constructed at another place with similar wind conditions.

- (d) Assume the wind speed at both turbines are independent of each other, construct a new transition matrix for the combined power output of both the turbines. Hint: you will need 10 states if you merge states where for example turbine 1 is in state 1 and turbine 2 is in state 2 with the state where turbine1 is in state 2 and turbine 2 is in state 1. Alternatively you can use 16 states and use the Kronecker Tensor product (under 7.1 in the course compendium).
- (e) Calculate the stationary distribution of this new Markov chain and find the probability that the combined 2 turbine system creates at least $400C$ power at any one point in time after a long time.