

MAA704, Perron-Frobenius theory and Markov chains.

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November 19, 2013

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Lecture overview

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Today we will look at:

- ▶ Permutation matrices and graphs.
- ▶ Perron frobenius for non-negative matrices.
- ▶ Stochastic matrices, Markov chains and their relation to Perron-Frobenius theory.
- ▶ Hitting times and hitting probabilities of Markov chain.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Repetition irreducible matrices.

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- ▶ A non-negative square matrix A is said to be **Irreducible** if any of the following equivalent properties is true:
- ▶ The graph corresponding to A is strongly connected.
- ▶ For every pair i, j there exists a natural number m such that $A_{i,j}^m \neq 0$.
- ▶ We have that $(I + A)^{n-1} > 0$.
- ▶ A cannot be conjugated into a block upper triangular matrix by a permutation matrix P .

Perron-
Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Permutation matrices.

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Perron-
Frobenius
theory and
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Definition

A **permutation matrix** is a square matrix with exactly one element equal to 1 in every row and column and 0 elsewhere.

- ▶ Permutation matrices are used to flip the position of individual rows or columns of a matrix.
- ▶ What happens when we multiply a permutation matrix with another square matrix?

Perron-
Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Permutation matrices.

- ▶ Multiplying P with A from the left permutes the rows of A according to the rows of P :

$$PA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ b & b & b & b \\ c & c & c & c \\ d & d & d & d \end{bmatrix} = \begin{bmatrix} b & b & b & b \\ c & c & c & c \\ a & a & a & a \\ d & d & d & d \end{bmatrix}$$

Permutation matrices.

- ▶ Multiplying P with A from the right permutes the columns of A according to the columns of P :

$$AP = \begin{bmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c & a & b & d \\ c & a & b & d \\ c & a & b & d \\ c & a & b & d \end{bmatrix}$$

Permutation matrices.

- ▶ What if we want to permute the same row and column of a matrix A according to the rows of P ?
- ▶ We multiply A with P from the left and with P^T from the right. Why P^T ?
- ▶ We need to multiply with P^T since multiplying with P from the right permutes according to the rows of P .

Permutation matrices and graphs

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If we multiply a graph A with a permutation matrix: PAP^T we only change the order of the nodes! The graph itself looks the same, only our matrix representation have changed.

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model

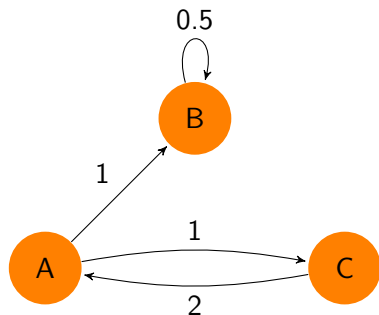
Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Permutation matrices and graphs



$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0.5 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow PAP^T = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

Both matrices represent the graph but with a different ordering of the vertices.

Perron-Frobenius for square non-negative irreducible matrices

Let A be a square non-negative irreducible matrix with spectral radius $\rho(A) = r$. Then we have:

1. r called the **Perron-Frobenius eigenvalue** is an eigenvalue of A , r is a positive real number.
2. The (right)eigenvector \mathbf{x} belonging to r called the (right)**Perron vector**, is positive ($A\mathbf{x} = r\mathbf{x}$, $\mathbf{x} > 0$).
3. The (left)eigenvector \mathbf{x} belonging to r called the (left)**Perron vector**, is positive ($\mathbf{x}^\top A = r\mathbf{x}^\top$, $\mathbf{x} > 0$).
4. No other eigenvector except those belonging to r have only positive elements.
5. The Perron-Frobenius eigenvalue r have algebraic and geometric multiplicity 1.
6. The Perron-Frobenius eigenvalue r satisfies:

$$\min_i \sum_j a_{i,j} \leq r \leq \max_i \sum_j a_{i,j}.$$

Perron-Frobenius for square non-negative irreducible matrices: Proof of some statements

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Frobenius
theory and
Markov
chains.

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Given (2) we can easily prove that the it's eigenvalue r is positive:

- ▶ If \mathbf{x} is a positive vector and A is non-negative non-zero (since it's irreducible) we have $A\mathbf{x} > 0$.
- ▶ Since $A\mathbf{x} = r\mathbf{x}$ if r is an eigenvalue belonging to the eigenvector \mathbf{x} we have $r > 0$.
- ▶ Since if r was negative, zero or complex $r\mathbf{x}$ would be as well, but $A\mathbf{x}$ is a positive vector.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

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Perron-
Frobenius
theory and
Markov
chains.

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We also prove (3) that the only positive eigenvectors is the ones belonging to r :

- ▶ Given A , $\rho(A) = r$ and $\mathbf{x} > 0$ is the (left) Perron vector of A .
- ▶ Assume we have a (right) eigenpair (λ, \mathbf{y}) , $\mathbf{y} \geq 0$.
- ▶ This gives: $\rho(A)\mathbf{x}^\top = \mathbf{x}^\top A$.
- ▶ $\Rightarrow \rho(A)\mathbf{x}^\top \mathbf{y} = \mathbf{x}^\top A\mathbf{y} = \mathbf{x}^\top \lambda \mathbf{y} \Rightarrow \rho(A) = \lambda$.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Perron-Frobenius for square positive matrices

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Perron-
Frobenius
theory and
Markov
chains.

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Initially the theorem was stated for positive matrices only and then later generalised for irreducible non-negative matrices. It's easy to see that if A is positive it's also non-negative and irreducible as well.

- ▶ If A is positive, in addition to all properties for irreducible non-negative matrices, the following is true as well:
- ▶ The Perron-Frobenius eigenvalue r have the strictly largest absolute value of all eigenvalues $|\lambda| < r$.

Perron-
Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Perron-Frobenius: Example permutation matrix

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Perron-
Frobenius
theory and
Markov
chains.

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We look at the permutation matrix:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- ▶ It's obviously non-negative and irreducible.
- ▶ From Perron-Frobenius we get that the spectral radius $\rho(A) = 1$ however the eigenvalues can easily be found to be ± 1 which are both on the unit circle.
- ▶ We let this last property of having or not having only one eigenvalue on the spectral radius divide the non-negative irreducible matrices in two different classes.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Perron-Frobenius: primitive matrices

We let A be a square non-negative irreducible matrix, then:

- ▶ A is **primitive** iff there is only one eigenvalue $r = \rho(A)$ on the spectral radius.
- ▶ A is **primitive** iff $\lim_{k \rightarrow \infty} (A/r)^k$ exists.

For the limit we have: $\lim_{k \rightarrow \infty} (A/r)^k = \frac{pq^\top}{q^\top p} > 0$.

Where p, q^\top is the Perron vectors for A, A^\top respectively.

- ▶ A is **imprimitive** iff there are $h > 1$ eigenvalues on the spectral radius. h is called it's **index of imprimitivity**.
- ▶ We note that for a matrix to be imprimitive it still needs to be irreducible.

Perron-Frobenius: Example permutation matrix

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Frobenius
theory and
Markov
chains.

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We return to the permutation matrix:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- ▶ Looking at the limit $\lim_{k \rightarrow \infty} (A/r)^k$ given $r = 1$.
- ▶ So P would be primitive if $\lim_{k \rightarrow \infty} (A)^k$ exists.
- ▶ However $P^2 = I$ and the sequence A, A^2, A^3, \dots alternate between P and I .

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Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Perron-Frobenius: primitive matrices

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Perron-
Frobenius
theory and
Markov
chains.

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Two ways to find if a non-negative irreducible matrix A is primitive is:

1. If any diagonal element $a_{i,i} > 0$ A is primitive.
2. A is primitive iff $A^m > 0$ for some $m > 0$.

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Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

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MAA704,
Perron-
Frobenius
theory and
Markov
chains.

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Engström

- ▶ If (1) is true it's obvious that there is a $m > 0$ such that $A^m > 0$.
- ▶ This since if we look at the graph: irreducibility guarantees there is a path from all nodes to all others and $A^m > 0$ means there is a path of exactly length m from all node to all others.
- ▶ However since we have one diagonal element $a_{i,i} > 0$ we can for all paths between two nodes going through $a_{i,i}$ increase the length of the path by one simply by looping within $a_{i,i}$ once.

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Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Perron-Frobenius: primitive matrices

MAA704,
Perron-
Frobenius
theory and
Markov
chains.

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- ▶ Now if $A^m > 0$ and A had a non positive eigenvalue $\lambda = \rho(A)$.
- ▶ Then λ^m should be an eigenvalue on the spectral radius of A^m .
- ▶ However since A^m is positive the only eigenvalue on $\rho(A^m)$ is positive and simple.

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Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Perron-Frobenius: imprimitive matrices

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Perron-
Frobenius
theory and
Markov
chains.

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We give a short note on imprimitive matrices as well:

- ▶ If A is a imprimitive matrix with h eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_h$ on the spectral radius $\rho(A)$. Then the following are true:
- ▶ $\text{alg mult}(\lambda_n) = 1, n = 1, 2, \dots, h.$
- ▶ $\lambda_1, \lambda_2, \dots, \lambda_h$ are the h th roots of $\rho(A)$, such that:

$$\{\lambda_1, \lambda_2, \dots, \lambda_h\} = \{r, r\omega, r\omega^2, \dots, r\omega^{h-1}\}, \quad \omega = e^{i2\pi/h}$$

Perron-
Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Application: Leontief Input-Output Model.

Suppose we have a closed economic system made up of n large industries, each producing one commodity. Let a J – unit be what industry J produces that sells for \$1.

If we have:

- ▶ $0 \leq s_j =$ number of J – units produced by industry j in one year.
- ▶ $0 \leq a_{ij} =$ number of I – units needed to produce one J – unit.
- ▶ $a_{ij}s_j =$ number of I – units consumed by industry J in one year.
- ▶ If we sum over all j we can then find the total I – units that are available to the public (d_i):

- ▶
$$d_i = s_i - \sum_{j=1}^n a_{ij}s_j.$$

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Perron-
Frobenius
theory and
Markov
chains.

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Suppose we have a public demand vector $\mathbf{d} = (d_1, d_2, \dots, d_n)$.
What is the minimal supply vector $\mathbf{s} = (s_1, s_2, \dots, s_n) \geq 0$
needed to satisfy a given demand $\mathbf{d} \geq 0$?

- ▶ We set up the linear system and think about the solution. Assuming it's solvable, do we need to make any further assumptions to guarantee the supply vector is non-negative $s \geq 0$?
- ▶ If we need to make some assumptions, are they realistic?

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Application: Leontief Input-Output model.

- ▶ We get the linear system $\mathbf{s} - A\mathbf{s} = \mathbf{d} \Rightarrow \mathbf{s} = (I - A)^{-1}\mathbf{d}$ if $(I - A)$ is non singular.
- ▶ Even though $A \geq 0$ we could end up with negative values in \mathbf{s} .
- ▶ If we assume $\rho(A) < 1$ we get $\lim_{k \rightarrow \infty} A^k = 0$, and the Neumann series $\sum_{n=0}^{\infty} A^n$ converges.
- ▶ If this converge we have $(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$ and \mathbf{s} will be non-negative since A, \mathbf{d} are both non-negative.

Application: Leontief Input-Output Model.

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Perron-
Frobenius
theory and
Markov
chains.

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- ▶ In order to guarantee that $\rho(A) < 1$ we can assume A is irreducible and use Perron-Frobenius for non-negative irreducible matrices.
- ▶ Since $\rho(A)$ is bounded by the maximum row or column sum (P-F) we can guarantee that $\rho(A) < 1$ if A is irreducible and the maximum column sum is less than one.
- ▶ This means $\sum_{n=0}^{\infty} A^n > 0$ converges (it's positive since A is irreducible).

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Application: Leontief Input-Output Model.

MAA704,
Perron-
Frobenius
theory and
Markov
chains.

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But are the assumptions realistic?

- ▶ Assuming A to be irreducible is the same as saying that every industry directly or indirectly uses something produced from all other industries.
- ▶ This seems like a reasonable assumption between large industries.
- ▶ The column sum c_j can be seen as the total number of units required to make one J – *unit*.
- ▶ It's seems reasonable to assume that an industry consumes commodities of a value less than one unit to produce a commodity with a value of one unit.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Stochastic matrices

We begin with the definition of a stochastic matrix.

- ▶ A row stochastic matrix is a non-negative matrix where each row sum to one.
- ▶ A column stochastic matrix is a non-negative matrix where each column sum to one.
- ▶ If it is both row and column stochastic we say that it is doubly stochastic.
- ▶ We will from now only say stochastic matrix when we in fact mean row stochastic matrix.
- ▶ Permutation matrices are doubly stochastic since only one element in every row and column is non-zero and that element is one.

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Frobenius
theory and
Markov
chains.

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Definition

A Markov chain is a random process with usually a finite number of states which continually transitions from one state to another. The transitions are memoryless in that they only depend on the current state and not on any previous state.

- ▶ Formal definition of a discrete time Markov chain:

$$\begin{aligned} P(X_{n+1} = x | X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) \\ = P(X_{n+1} = x | X_n = x_n) \end{aligned}$$

- ▶ Every stochastic matrix defines a Markov chain and every Markov chain defines a stochastic matrix.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

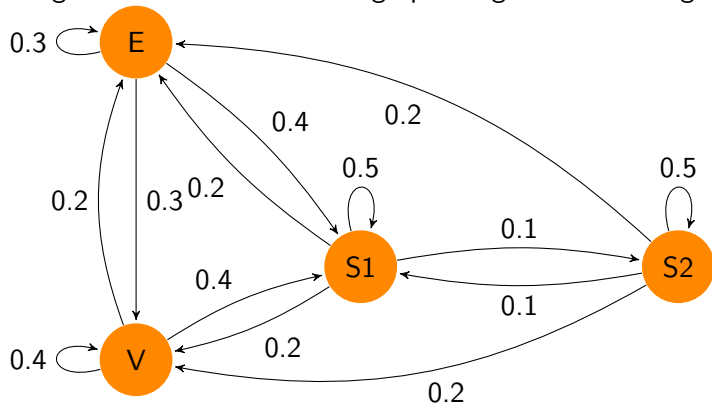
Markov chains: Car rental service

Consider a car rental service with 4 stores where you can rent a car in one store and leave it in another. There is one store in Västerås, one in Eskilstuna and two in Stockholm (1 and 2). Estimates of the transition probabilities for where a car is left at the end of the day, depending on where it started the day is:

- ▶ A car rented in Västerås is left with probability 0.4 in Västerås, 0.4 in Stockholm 1 and 0.2 in Eskilstuna.
- ▶ A car rented in Eskilstuna is left with probability 0.3 in Eskilstuna, 0.4 in Stockholm 1 and 0.3 in Västerås.
- ▶ A car rented in any of the shops in Stockholm is left with probability 0.2 in Västerås and Eskilstuna, 0.1 in the other shop in Stockholm and 0.5 in the same shop in Stockholm.
- ▶ This is quite hard to follow in text, but since it's a markov chain we can write its graph and stochastic matrix.

Markov chains: Car rental service

Writing the model as a directed graph we get the following:



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Perron-
Frobenius
theory and
Markov
chains.

Christopher
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With corresponding transition matrix (distance matrix) P :

$$\begin{pmatrix} \text{Eskilstuna} \\ \text{Västerås} \\ \text{Stockholm 1} \\ \text{Stockholm 2} \end{pmatrix} \quad P = \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.2 & 0.4 & 0.4 & 0 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$$

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Markov chains: Car rental service

Since there is only a service station at the shop in Västerås we want to estimate how often a car is there. We consider a car starting in Eskilstuna.

- ▶ If we want to know the probability that the car is in Västerås tomorrow we simply multiply our initial distribution $X_0^T = [1 \ 0 \ 0 \ 0]$ with P and look at the corresponding element.

$$X_1^T = X_0^T P = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.2 & 0.4 & 0.4 & 0 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \\ 0 \end{bmatrix}^T$$

- ▶ And we see there is a probability 0.3 that it will be in Västerås tomorrow.

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Perron-
Frobenius
theory and
Markov
chains.

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Engström

If we want to know the probability for the car to be in Västerås the day after tomorrow, simply repeat the procedure on X_1 :

$$X_2^T = X_1^T P = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.2 & 0.4 & 0.4 & 0 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.23 \\ 0.29 \\ 0.44 \\ 0.04 \end{bmatrix}^T$$

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Stochastic matrices: Markov chains

MAA704,
Perron-
Frobenius
theory and
Markov
chains.

Christopher
Engström

To get the probability distribution π_k of the cars on day k we calculate:

$$\pi_k = X_0^T P^k$$

- ▶ One Important question in the theory of Markov chains is wheather the limit $\pi = \lim_{k \rightarrow \infty} X_0^T P^k$ exists.
- ▶ This can be seen as the distribution of the time the Markov chain spends in every state.
- ▶ We also want to know if it converges towards the same distribution π regardless of initial state X_0 .

Perron-
Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Stochastic matrices: Markov chains

MAA704,
Perron-
Frobenius
theory and
Markov
chains.

Christopher
Engström

In order to answer these questions we divide the stochastic matrices in 4 different groups:

1. Irreducible with $\lim_{k \rightarrow \infty} (P^k)$ existing. (P is primitive)
2. Irreducible with $\lim_{k \rightarrow \infty} (P^k)$ not existing. (P is imprimitive)
3. Reducible and $\lim_{k \rightarrow \infty} (P^k)$ existing.
4. Reducible and $\lim_{k \rightarrow \infty} (P^k)$ not existing.

Perron-
Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Irreducible stochastic matrices

MAA704,
Perron-
Frobenius
theory and
Markov
chains.

Christopher
Engström

If P is primitive we can immediately use Perron-Frobenius.

- ▶ For P we get the (right) Perron vector \mathbf{e}/n as the unit vector divided by the number of elements n .
- ▶ If we let $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ be the (right) Perron vector for P^\top we get the limit:



$$\lim_{k \rightarrow \infty} P^k = \frac{(\mathbf{e}/n)\boldsymbol{\pi}^\top}{\boldsymbol{\pi}^\top(\mathbf{e}/n)} = \mathbf{e}\boldsymbol{\pi}^\top > 0$$

- ▶ Calculating the probability distribution of the limit gives:
 $\lim_{k \rightarrow \infty} X_0^\top (P^k) = X_0^\top \mathbf{e}\boldsymbol{\pi}^\top = \boldsymbol{\pi}^\top$.
- ▶ The Markov chain converges to the (right) Perron vector of P^\top regardless of initial state X_0 .

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Stochastic matrices: Markov chains

We return to the Markov chain for our model for the car rental service:

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0.2 & 0.4 & 0.4 & 0 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$$

- ▶ It's easy to see that there is a path from any state to any other so P is obviously irreducible.
- ▶ Since at least one diagonal element is non-zero it's primitive as well and we know that the limit exist:

$$\lim_{k \rightarrow \infty} X_0^T P^k = X_0^T \mathbf{e} \boldsymbol{\pi}^T = \boldsymbol{\pi}^T$$

- ▶ Calculating the Perron vector $\boldsymbol{\pi}^T$ yields $\boldsymbol{\pi}^T = [0.2222 \ 0.2778 \ 0.4167 \ 0.0833]^T$.
Which is the stationary distribution of our Markov chain.

Irreducible stochastic matrices

If P is imprimitive we know that it has $h > 1$ simple eigenvalues on the spectral radius $\rho(P) = 1$.

- ▶ We also still have the (right) Perron-vector \mathbf{e}/n for P .
- ▶ While the limit $\lim_{k \rightarrow \infty} P^k$ doesn't exist, we can compute the Cesàro sum:

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^k P^j$$

- ▶ This converges in the same way as in the primitive case and we get:

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^k P^j = \mathbf{e}\boldsymbol{\pi}^T > 0$$

Irreducible stochastic matrices

We multiply the limit with the initial vector X_0 and get just like in the primitive case:



$$X_0 \left(\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^k P^j \right) = X_0^T \mathbf{e} \boldsymbol{\pi}^T = \boldsymbol{\pi}^T$$

- ▶ Although the stationary distribution $\boldsymbol{\pi}$ exist regardless of the choice of X_0 the Markov chain doesn't converge to it but rather oscillate around it.

Stochastic matrices: Markov chains

We return to our small permutation matrix:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- ▶ We already saw that it's imprimitive and therefore the limit doesn't exist.
- ▶ However limit of the Cesàro sum exist and we still get the stationary distribution π^\top as the Perron vector.
- ▶ This gives the stationary distribution $\pi^\top = [0.5 \ 0.5]^\top$.
- ▶ So while the Markov chain won't converge it will on average spend half the time in state 1 and half the time in state 2.
- ▶ In fact we can easily see that it spends exactly every second step in state 1 and the other in state 2.

Reducible stochastic matrices

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Frobenius
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If the stochastic matrix is reducible we can't directly apply Perron-Frobenius.

- ▶ What we can do is to get as close as possible to the irreducible case.
- ▶ We recall that a reducible matrix A could be permuted into upper triangular form by a permutation matrix P :

▶

$$PAP^T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Reducible stochastic matrices

- ▶ If X or Z is reducible as well we can permute it further until we end up with:



$$PAP^T = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ 0 & X_{22} & \cdots & X_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & X_{kk} \end{bmatrix}$$

- ▶ Where X_{ii} is either irreducible or a single zero element.

Reducible stochastic matrices

- ▶ Last we look for rows of blocks where only the diagonal block X_{jj} contains non-zero elements and permute those to the bottom.

$$\left[\begin{array}{cccc|cccc} X_{11} & X_{12} & \cdots & X_{1j} & X_{1,j+1} & X_{1,j+2} & \cdots & X_{1,k} \\ 0 & X_{22} & \cdots & X_{2j} & X_{2,j+1} & X_{2,j+2} & \cdots & X_{2,k} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & X_{jj} & X_{j,j+1} & X_{j,j+2} & \cdots & X_{j,k} \\ \hline 0 & 0 & \cdots & 0 & X_{j+1,j+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & X_{j+2,j+2} & \ddots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X_{kk} \end{array} \right]$$

- ▶ We note that it's still only essentially a re-labeling of the vertices.

Reducible stochastic matrices

- ▶ This is called the **canonical form** of the matrix.
- ▶ We call the states in the upper left the **transient** states. Whenever we leave one of them, we can never come back.
- ▶ In the same way we call the ones in the bottom right the **ergodic** or **absorbing** states. Every one of these blocks is a irreducible stochastic matrix in itself.

Reducible stochastic matrices

Assuming the reducible stochastic matrix A is written in the canonical form.

$$A = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

- ▶ It's possible to find the limit $\lim_{k \rightarrow \infty} A^k$ if all blocks in Z are primitive matrices or the corresponding limit of the Cesáro sum if at least one of them are imprimitive.
- ▶ This limit can be written (primitive case):

$$\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} 0 & (I - X)^{-1}YZ \\ 0 & E \end{bmatrix}$$
$$E = \begin{bmatrix} \mathbf{e}\pi_{j+1}^\top & 0 & \cdots & 0 \\ 0 & \mathbf{e}\pi_{j+2}^\top & \ddots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathbf{e}\pi_k^\top \end{bmatrix}$$

Reducible stochastic matrices

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Perron-
Frobenius
theory and
Markov
chains.

Christopher
Engström

- ▶ The elements $a_{p,q}$ of $(I - X)^{-1}Y$ holds the probability that starting in p we leave the transient states and eventually hit state q in one of the absorbing blocks. This is commonly called the hitting probability to hit q starting in p .
- ▶ The vector $(I - X)^{-1}\mathbf{e}$ is of interest as well. It holds the average number of steps needed to leave the transient state, we call this the hitting times for reaching the absorbing states.

Perron-
Frobenius

Leontief
Input-Output
model

IMarkov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Car rental service, hitting times

MAA704,
Perron-
Frobenius
theory and
Markov
chains.

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While on average one of our cars is in Västerås about 0.28% of the time from looking at the stationary distribution.

We don't know how long we can expect to wait in between visits to Västerås (and the service station).

- ▶ We seek to find how many days on average it takes before a car ends its day in Västerås.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Car rental service, hitting times

To find this we make a small adjustment to the Markov chain by making the state representing Västerås absorbing (only linking to itself).

- ▶ Then we could use what we learned of reducible Markov chains to find the expected hitting time.
- ▶ We get new transition matrix:

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0 & 1 & 0 & 0 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$$

Car rental service, hitting times

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Perron-
Frobenius
theory and
Markov
chains.

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Now if we could write P in canonical form, then we could calculate $(I - X)^{-1}\mathbf{e}$ where X is the top left part of our new matrix P and use this to find the hitting times.

- ▶ To write it in canonical form we only need the state representing Västerås to be in the bottom right corner of P .
- ▶ We permute P using permutation matrix M which we want to swap the position of the second and forth row/column.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Car rental service, hitting times

- This gives: $MPM^T =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0 \\ 0 & 1 & 0 & 0 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

►

$$= \begin{bmatrix} 0.3 & 0 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.5 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \text{Eskilstuna} \\ \text{Stockholm 2} \\ \text{Stockholm 1} \\ \text{Västerås} \end{pmatrix}$$

Car rental service, hitting times

- ▶ This gives:

$$X = \begin{bmatrix} 0.3 & 0 & 0.4 \\ 0.2 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.5 \end{bmatrix}$$

- ▶ And hitting times $(I - X)^{-1}\mathbf{e}$

$$= \begin{bmatrix} 0.3 & 0 & 0.4 \\ 0.2 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4.5 \\ 4.5 \end{bmatrix} \begin{pmatrix} \text{Eskilstuna} \\ \text{Stockholm 2} \\ \text{Stockholm 1} \end{pmatrix}$$

- ▶ And we can see for example that it takes on average 4 days until a car in Eskilstuna ends the day in Västerås.

Application: Return to the Leontief model

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We return to the Leontief Input-Output model.

- ▶ We remember we got a linear system $\mathbf{s} - A\mathbf{s} = \mathbf{d}$.
- ▶ What properties or restrictions of A guarantee that we can satisfy any demand \mathbf{d} ?
- ▶ Look at the system as a Markov chain with transition matrix A
- ▶ We introduce a new absorbing state D corresponding to the demand such that $A_{i,d} = 1 - \sum_j a_{ij}$.
- ▶ When does the Markov chain always converge to the absorbing state D ?
- ▶ Can you find the gross national product (GNP) in relation to the Markov chain given a demand \mathbf{d} ?

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Application: Return to the Leontief model

- ▶ We already saw that A irreducible with $\rho(A) < 1$ is enough.
- ▶ We let A represent the part consisting of the transient states in a Markov chain as discussed in the problem.
- ▶ If A consists of transient states, every initial supply \mathbf{s} leads to supplying some demand \mathbf{d} .
- ▶ Otherwise the state wouldn't be able to transition into the absorbing state.
- ▶ The same is needed for the Markov chain to always converge. If every industry is either making a profit or rely on another industry making a profit, there is a path to the absorbing state.

Application: Return to the Leontief model

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Perron-
Frobenius
theory and
Markov
chains.

Christopher
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- ▶ GNP is the total supply \mathbf{s} produced to both satisfy the demand as well as the internal demand of the industry.
- ▶ If we look at the mean number of steps required to reach the absorbing state from the transient states:
$$T = (I - A)^{-1}\mathbf{e}.$$
- ▶ We can see the elements t_i as the expected production in units to create one product unit of industry i .
- ▶ Multiplying the demand vector \mathbf{d} with T : $(T^T \mathbf{d})$ gives the expected total produced units. i.e. the GNP.

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model

Next lecture:

- ▶ Be sure to choose group and subject for the projects before the next lecture this friday.
- ▶ Next lecture will be about Matrix factorizations and canonical forms.

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Perron-
Frobenius
theory and
Markov
chains.

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Engström

Perron-
Frobenius

Leontief
Input-Output
model

Markov
chains

Reducible
stochastic
matrices

Car rental
service, hitting
times

Return to
Leontief model