# Mathematics behind Internet, MAA507

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# L2: Incidence matrix, Laplacian matrix and applications

Todays lecture:

- Incidence matrix.
- Laplacian matrix.
- Minimum spanning trees.

# Incidence matrix

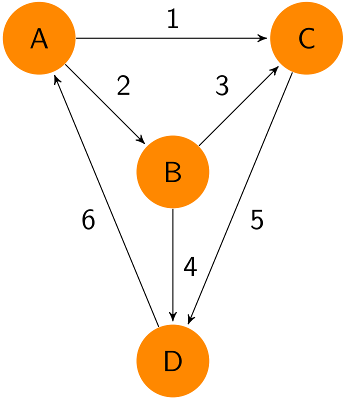
## Definition

The **incidence matrix** $B$ of a directed graph with no loops $G$ with vertices $v_1, v_2, \ldots v_n$ and edges $e_1, e_2 \ldots e_m$ is a $m \times n$ matrix with elements $b_{ij}$ such that:

$$b_{ij} = \begin{cases} 1, & \text{if } e_i \text{ points towards } v_j \\ \text{-1}, & \text{if } e_i \text{ starts in } v_j \\ 0, & \text{if } e_i \text{ neither starts or ends in } v_j \end{cases}$$

# Incidence matrix

Incidence matrix of a directed graph, with edges numbered $1, 2, \ldots, 6$.




$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

# Incidence matrix

The incidence matrix have applications in:

- Calculating voltages and electric currents in electrical networks.
- Flows in pipeline networks.
- Transportation networks.

# Incidence matrix and electrical networks



**Kirchhoff's current law**: (KCL) The sum of all currents flowing into a node is equal to the sum of all currents flowing out of that node.

$$\sum_{k=1}^{n} I_k = 0$$

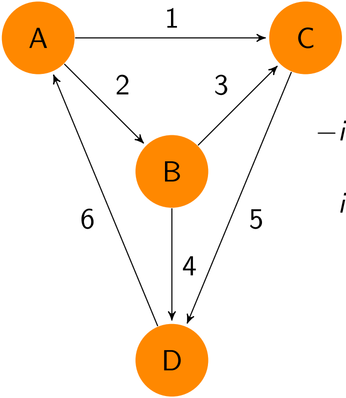
# Incidence matrix and electrical networks

**Kirchhoff's voltage law**: (KVL) The voltage over an edge is equal to the difference in potential between it's start and end node.
Can also be stated as: The directed sum of voltage around any closed network is zero:

$$\sum_{k=1}^{n} V_k = 0$$

# Incidence matrix and electrical networks

Using Kirchoff's laws we can set up linear equation systems for the currents, voltages and potentials in a network.
From Kirchhoff's current law we get using our example:




$$
\begin{array}{rrrrrrl}
-i_1 & -i_2 & & & & i_6 & = 0 \\
 & i_2 & -i_3 & -i_4 & & & = 0 \\
i_1 & & i_3 & & -i_5 & & = 0 \\
 & & & i_4 & i_5 & -i_6 & = 0
\end{array}
$$

# Incidence matrix and electrical networks



Written on matrix form $B^\top I = 0$:

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Incidence matrix and electrical networks

Similarily we get a linear system using Kirchoff's voltage law:

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_A \\ p_B \\ p_C \\ p_D \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

# Tellegens theorem

One useful theorem for electrical networks we can now easily prove is Tellegens theorem:

## Theorem

**Tellegens theorem**: *Assume* $I = (i_1, i_2, \ldots, i_n)$ *is currents measured over the edges of a network measured at the same time and* $v = (v_1, v_2, \ldots, v_n)$ *be voltages measured over the edges of the same network measured at the same time (but not neccesary at the same time as the currents). Then:*

$$v^\top I = v_1 i_1 + v_2 i_2 + \ldots + v_n i_n = 0$$

# Tellegens theorem

Proof.

$$v^\top I = (Bp)^\top I = (p^\top B^\top) I = p^\top B^\top I = p^\top 0 = 0$$

□

# Tellegens theorem

Kirchoff's theorem and Tellegens theorem hold for many other networks other than electrical.
Here are some examples and their corresponding variables $av(t), tv(t)$

- Electrical networks: $av(t) = v(t), tv(t) = i(t)$.
- Flow networks:
  $av(t) =$ pressure difference $p(t), tv(t) =$ flow rate $q(t)$.
- Mechanical translation:
  $av(t) =$ velocity $v(t), tv(t) =$ force $f(t)$.
- Thermal systems:
  $av(t) =$ degree difference $\theta^\circ\mathrm{C}, tv(t) =$ heat flow $\phi(t)$.

# Incidence matrix and electrical networks



The equations from Kirchoff's theorems are not linearly independent.

- Given for example the voltages $v$ we cannot find the potentials $p$.
- We can however often solve them by grounding one node as seen later.

# Rank-nullity theorem

### Theorem

**Rank-nullity theorem**: *If $A$ is a $m \times n$ matrix then:*

$$rank\ A + nullity\ A = n$$

- rank $A$ is the number of linearly independent column vectors of $A$.
- nullity $A$ is the dimension of the nullspace:

$$\{x \in \mathbb{R}^n | Ax = 0\}$$

# Rank-nullity theorem

### Theorem

*nullity A of the incidence matrix of a directed graph is equal to the number of connected components.*

- In other words, for a connected graph we can for example ground one node and then find the potentials given the voltages.

# Rank-nullity theorem

Grounding node $D$ for our example gives:

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_A \\ p_B \\ p_C \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

# Rank-nullity theorem

This gives:

$$\begin{cases} v_1 + v_5 + v_6 = 0 \\ v_2 + v_4 + v_6 = 0 \\ v_3 - v_4 + v_5 = 0 \\ p_B = -v_4 \\ p_C = -v_5 \\ p_A = v_6 \end{cases}$$

The first three are cycles and should be zero from Kirchoff's voltage law.

# Laplacian matrix

The **Laplacian matrix** $L$ of a graph with no self loops is defined as:

$$L := D - A$$

where $D$ is the **degree matrix** and $A$ is the **adjacency matrix**. This gives elements $l_{i,j} \in L$:

$$l_{i,j} := \begin{cases} \deg(i), & i = j \\ -1, & i \neq j, v_i \text{ links to } v_j \\ 0, & \text{otherwise} \end{cases}$$

Where $deg(i)$ is the number of edges connected with vertice $v_i$.

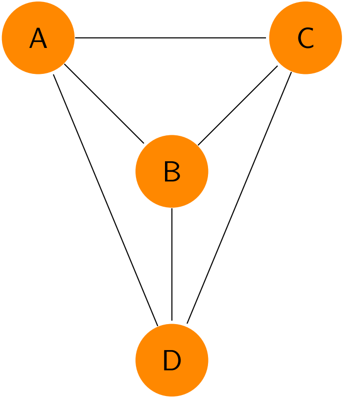
# Laplacian matrix

- For a directed graph we can use either indegree or outdegree depending on application.
- The laplacian matrix can also be written using the incidence matrix $B$:

$$L = B^\top B$$

This yields the Laplacian matrix using the adjacency matrix of the undirected graph.

# Laplacian matrix




$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

# Laplacian matrix

A couple of properties of the Laplacian matrix $L$ of undirected graphs.

- $L$ is a positive semidefinite matrix ($z^\top L z \geq 0$)
- All eigenvalues of $L$ is non-negative.
- $\lambda_1 = 0$ is an eigenvalue and the number of connected components is it's algebraic multiplicity.
- The smallest non zero eigenvalue is called the spectral gap.
- The second smallest eigenvalue of $L$ is the algebraic connectivity of the graph.

# Laplacian matrix



Some applications of the Laplacian matrix is:

- Using Kirchhoff's theorem: Find the number of spanning trees in a graph.
- Calculate the effective resistance between points in a electrical network.
- Using Cheeger's inequality: Estimate whether the network have "bottlenecks" or is well connected.

# Cheeger constant

## Definition

Given a undirected graph $G$ with vertices $V$ and edges $E$. Let $A \subseteq V$ be some of the vertices in $G$, and $\partial A$ be the edges from $A$ to vertices outside of $A$.

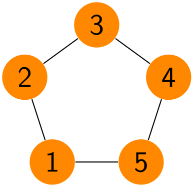
$$\partial A := \{(x, y) \in E | x \in A, y \in V \setminus A$$

The **Cheeger constant** $h(G)$ is then:

$$h(G) := \min \left\{ \frac{|\partial A|}{|A|} \middle| A \subseteq V, 0 < |A| \leq \frac{|V|}{2} \right\}$$

# Application: Robustness of a computer network.

We look at ring networks such as the one below:



# Application: Robustness of a computer network.

- We divide the set of vertices $V = \{v_1, v_2, \dots, v_n$ in two equally large groups:

$$A = \{v_1, v_2, \dots, v_{\lfloor n/2 \rfloor}\}$$

- We have only two edges connecting the two sets:

$$\partial A = \{(v_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor} + 1), (v_n, v_1)\}$$

- And we get Cheeger constant:

$$h(G) = \frac{|\partial A|}{|A|} = \frac{2}{\lfloor n/2 \rfloor} \to 0, n \to \infty$$

# Application: Robustness of a computer network.

While we haven't proved that this is the true minimum, we have proved that it is at least this small, and goes towards zero as the number of computers increase.

- A low Cheeger constant as we have here means that we have a highly bottlenecked network, something which is obvious in that if only 2 computers fail, we could end up with a disconnected network.
- A high Cheeger constant would instead mean that we have a robust network.
- The Cheeger constant is especially important when you have a graph structure and whant to see what would happen if you expand it by increasing the number of nodes.

# Application: Robustness of a computer network.

The Laplacian matrix can be used to give a bound on the Cheeger constant:

## Theorem

**Cheeger's inequality** *Given a graph $G$ with Laplacian matrix $L$ with the smallest positive eigenvalue $\lambda_1$ and Cheeger constant $h(G)$ the following inequality holds:*

$$\lambda_1 \geq \frac{h^2(G)}{4}$$

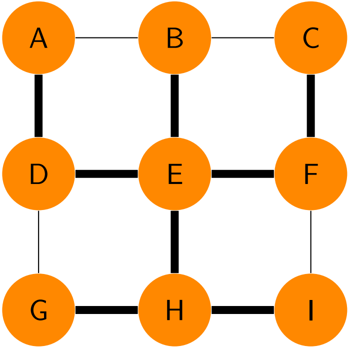
# Spanning tree

## Definition

Given a connected undirected graph $G$. A **spanning tree** $T$ is a set of edges of $G$ that connecs all vertices but have no cycles or loops.

- We can consider a spanning tree a set of edges which is enough to create a connected graph connecting all the original vertices.
- Adding a new edge to a spanning tree always creates a cycle (so that it's not a spanning tree anymore).

# Spanning tree




A spanning tree connected by **bold** edges in a grid graph.

# Spanning tree



If we have a weighted graph we are often interested in the spanning tree with the least total weight.

## Definition

Given a connected undirected weighted graph $G$. A **minimum spanning tree (MST)** is the spanning tree with the least total weight in the graph.

- If the graph is not connected we sometimes look for a **minimum spanning forest** which is composed of minimum spanning trees for every connected component.

# Minimum spanning tree

Some applications related to finding a minimum spanning tree are:

- Hierarchical clustering using single-linkage clustering (minimum distance between clusters is the minumum distance between any pair of nodes, one from each cluster).
- Image segmentation, (covered on seminars)
- Broadcasting in computer networks.
- Creating taxonomy trees.
- And much more.

# Application, resistance-distance



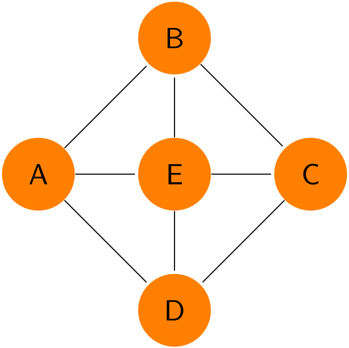
The resistance distance between two nodes $v_i$, $v_j$ in a graph $G$ is defined as the effective resistance between them when each edge is replaced with a 1 Ω resistor.

- We define: $\Gamma = L + \frac{1}{n}$.
- Elements of the resistance distance is then:

$$(\Omega)_{ij} = \Gamma_{ii}^{-1} + \Gamma_{jj}^{-1} - 2\Gamma_{ij}^{-1}.$$

- This is the Moore-Penrose pseudoinverse of the Laplacian.
- Resistance-distance is a Metric on graphs.

# Application, resistance-distance




$$L = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

Figure: Undirected graph and corresponding Laplacian matrix

# Application, resistance-distance

We get $\Gamma = L + 1/5$ and it's inverse:

$$\Gamma^{-1} = \begin{bmatrix} 0.4267 & 0.1600 & 0.0933 & 0.1600 & 0.1600 \\ 0.1600 & 0.4267 & 0.1600 & 0.0933 & 0.1600 \\ 0.0933 & 0.1600 & 0.4267 & 0.1600 & 0.1600 \\ 0.1600 & 0.0933 & 0.1600 & 0.4267 & 0.1600 \\ 0.1600 & 0.1600 & 0.1600 & 0.1600 & 0.3600 \end{bmatrix}$$

# Application, resistance-distance

$$(\Omega) = \begin{bmatrix} 0 & 0.5333 & 0.6667 & 0.5333 & 0.4667 \\ 0.5333 & 0 & 0.5333 & 0.6667 & 0.4667 \\ 0.6667 & 0.5333 & 0 & 0.5333 & 0.4667 \\ 0.5333 & 0.6667 & 0.5333 & 0 & 0.4667 \\ 0.4667 & 0.4667 & 0.4667 & 0.4667 & 0 \end{bmatrix}$$

# Application, resistance-distance



## Theorem

**Matrix-tree theorem**: *Given a connected graph $G$ with $n$ vertices, laplacian matrix $L$ and non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$. The number of spanning trees is:*

$$t(G) = \frac{1}{n}\lambda_1\lambda_2\ldots\lambda_{n-1} = det\ L(i)$$

*Where $L(i)$ is the submatrix optained by deleting row and column $i$.*

# Application, resistance-distance

Let $G$ be a graph with vertices $V$, edges $E$ and laplacian matrix $L$. The resistance distance is related to the number of spanning trees as follows:

$$(\Omega)_{i,j} = \frac{\det L(i,j)}{\det L(i)} = \frac{\det L(i,j)}{t(G)}$$

Where $L(i)$ is the submatrix optained by deleting row and column $i$ and $L(i,j)$ is the submatrix optained by deleting row $i$ and $j$ and column $i$ and $j$.