Seminar assignment 1

This is the alternate version of seminar assignment 1 that is intended for reserves that were accepted to the course after the deadline of seminar assignment 1 or those who wish to improve their results from assignment 1.

Each seminar assignment consists of two parts, one part consisting of $3-5$ elementary problems for a maximum of 10 points from each assignment. For the second part consisting of problems in applications you are to choose one of three problems to solve. This part can give up to 5 points from each assignment. Each seminar assignment can give a maximum of 15 points, to pass you will need at least 20 points total from both the assignments.

The first part consists of elementary questions to make sure that you have understood the basic material of the course while the second part consists of one larger application example.

Solutions can either be handed in by email to *christopher.engstrom@mdh.se* or *karl.lundengard@mdh.se* or you can hand in handwritten solutions either during the lectures or in the box outside Christopher and Karls room $U3:185$.

Solutions should be submitted by the 14th of January at 23:59.

1 Part 1

In the first part you are to solve and hand in solutions to the questions. You are allowed to use computer software to check your results, but your calculations as well as your result should be included in the answers for full points.

1.1

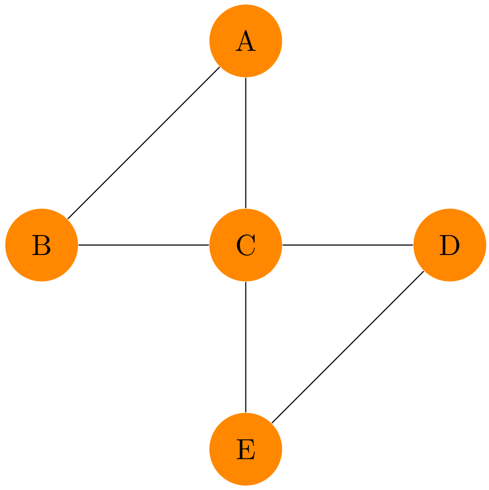
- (1) Define when two matrices are called *similar*.
- (1) Use the characteristic polynomial $\det(I\lambda - A)$ to find the eigenvalues of:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 2 & -2 \\ 0 & 3 & -2 \end{bmatrix}$$

1.2

Consider the graph:

- (1) What is the Adjacency matrix of the graph.
- (1) What is the Laplacian matrix of the graph.



1.3

We consider the Markov chain on the states (s_1, s_2, s_3, s_4) given by the stochastic matrix P below:

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 2/3 & 1/3 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- (1) Given the stochastic matrix above, what is the graph of the corresponding Markov chain?
- (1) If we start in state $X_0 = s_1$ with a probability 0.5 and in s_4 with a probability 0.5 $(P(X_0 = s_1) = (P(X_0 = s_4) = 0.5$. What is the probability to be in state s_2 after 1 step?
- (1)Use the memoryless property of Markov chains to find the probability to be in s_3 at step 6 if we were in s_3 at step 3, $(P(X_6 = s_3|X_3 = s_3))$.

1.4

- (1) Write the definition of a irreducible matrix and give a short example how you can show that a matrix is irreducible.

Consider the matrix A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \\ 5 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is irreducible (you do not have to show this), use Perron-Frobenius theorem and answer:

- (1) Without calculating the eigenvalues, use Perron-Frobenius to give a bound for the largest and smallest possible value of the Perron-Frobenius eigenvalue.
- (1) Without calculating the eigenvalues, using Perron-Frobenius: Which of the following values could be eigenvalues of A?

$$x_1 = 7, \ x_2 = -5i, \ x_3 = 2, \ 4 - 4i$$

  (Note that we are only interested in which values that could be eigenvalues, not which values that actually are eigenvalues of this matrix).

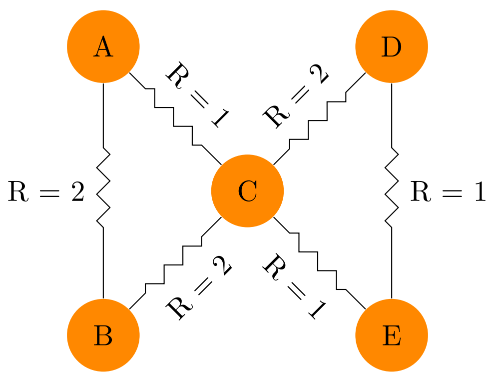
2 Part 2

In this second part you are to choose **ONE** example where you attempt to solve the questions presented. If you hand in answers to more than one choice you will get points corresponding to the choice which would give the least total points.

You are allowed to use computer software (but it shouldn't be needed) to solve the questions as long as you make it clear what you are doing and give an example of what could be used to solve it by hand. For example you could write "I solved the linear system $Ax = b$ using Matlabs "fsolve", another method would have been to use Gaussian elimination and solving the found triangular system".

2.1 Voltage and current in electrical networks

Assuming we have a connected graph with edges corresponding to resistors and assign a fixed current v_a to vertice a and $v_b = 0$ to vertice b. Our goal is to use a Markov chain approach to find the voltage v_i of the other vertices as well as the current I_{ij} along the edges of the graph.



We look at the graph above where we put a 3-volt battery across points A and D such that we get a voltage $v_a = 3,\ v_d = 0$. To find the voltage we will look at a random walk on the graph where A, B are absorbing states and the transition probabilities for all other vertices i are decided by the resistors such that.

$$P_{ij} = \frac{C_{ij}}{\sum_{j=1}^{n} C_{ij}}, i \neq A, B$$

$$C_{ij} = \frac{1}{R_{ij}}$$

- (2) Contruct the new graph and corresponding transition matrix where the transition probabilities are given above. Remember that state A and D are set as absorbing states.

Since we have absorbing states, the Markov chain is reducible and can be brought to upper block triangular form.

- (1) Find a permutationmatrix P which can be used to bring your transitionmatrix to upper block triangular form. Use your P to bring your transition matrix to upper block triangular form.

We then look at the random walk on the graph starting in vertice i. The voltage v_i can then be seen as the probability that we end up in A rather than D after absorbtion in one of the states A, D. In other words we seek the hitting probabilities h_i

- (1) Use your transition matrix brought to upper block triangular form to find the hitting probabilities h.

The voltage is calculated as $v_i = v_a h_i$.

The current along an edge I_{ij} can be found as the expected number of times to go from i to j when we start in a and only let b be an absorbing state. This however can also be found using Ohm's Law:

$$I_{ij} = \frac{v_i - v_j}{R_{ij}}$$

(Note that $I_{ij} = -I_{ji}$)

- (1) Calculate the current over the edges and write the graph, noting currents with their value and direction on the edges and voltages on the vertices.

2.2 Credit rating of bonds

We consider how the credit rating of bonds change from one rating to another or defaulting. The bonds are rated once every year at the same time at the end of the year. From historical data we have found that the credit rating of a bond can be modelled using a markov chain with transition matrix P. We consider the 3 ratings A, B, C and default D. We note that this is a simplified model in order to allow calculations by hand, in reality there should be a possibility to move between $A \leftrightarrow D$ as well as there possibly being more ratings such as A, AA, AAA.

To work with we have some data from only a single year, where we have how many that start every year at every rating A, B, C in column S. And how those same bonds are rated at the end of the year in columns R_A, R_B, R_C, R_D.

	S	R_A	R_B	R_C	R_D
A	75	70	4	1	0
B	100	5	85	5	5
C	25	0	4	15	6

We estimate the transition probabilities of a Markov chain describing the change in ratings of a bond as:

$$P_{ij} = \frac{R_{ij}}{S_i}$$

Where R_{ij} is the number of bonds that ended up in j starting in i at the beginning of the year and S_i is the number of bonds starting in i at the beginning of the year.

- (1) From the table above: Estimate the transition probabilities of a Markov chain describing the change of rating of a bond over time using the above estimate. Use these to create the transition matrix for this Markov chain. Note that we should have 4 states, with the last one "default" D absorbing.

Often we are interested in not only how likely a bond is to say default during the next year, but also over many years. While it's true that a bond rated C seems to be more likely to default than a bond rated B in one year. The same might not be true if we consider the bond over 3 years.

- (1) What is the probability that a C rated bond will be in default after 2 years? ($P(X_3 = D|X_0 = C)$).

While it's obvious that all bonds in our model will eventually default, we are interested in how long we can expect it to take. We remember we could find the hitting time k_i as the minimal solution to:

$$\begin{cases} k_i^A = 0, & i \in A \\ k_i = 1 + \sum_{j=1}^{n} p_{ij} k_j^A & i \notin A \end{cases}$$

- (1) Find the expected time until default for a bond starting in B, (k_B). You can either set up the linear system $k = Ak + c$ to solve it on the computer, or solve the system by hand.

The model above have one problem when we want to look at all the bonds, every bond is bound to eventually default and since we never introduce new bonds, eventually all bonds will have defaulted. But we want to say something about the long term as well. We consider the model where we always have the same number of bonds. We do this by creating a new bond every time one defaults. At the same time we remove the defaulting bond since it is for us no longer interesting. We let the rating of the new bond be distributed in the same way as our original distribution of bonds at the beginning of the year.

- (1) Modify the transition matrix by changing the row corresponding to the previously absorbing state as discussed above.

Now we can talk about the long term behaviour of the chain as well as answer questions such as, how large percentage of total bonds can we expect to be A rated.

- (1) What is the stationary distribution of this new Markov chain? Assuming a constant 200 bonds, how many can we expect to default every year after a long time?

2.3 Fail-Safe system

Consider a machine with two independent controls A, B designed to immediately stop the machine if something bad is about to happen. Whenever the machine is stopped by A or B the problem can quickly be fixed and the machine restarted almost immediately. However if both controls fails to stop the machine at the same time something bad have happened to either the machine or the operator. We consider the discrete time points $t_1, t_2, \ldots$ where the machine needs to be stopped.

If one control fails to activate it is immediately replaced with a new one before the next time point. A control that worked during the last time of activation t_n have a 90% chance to work at the next time it needs to activate t_{n+1}. However immediately after a control fails, the new untested replacement have only a 75% to activate at the next time step, if it works correctly in that step it's considered "tested" and works with 90% reliability after that.

We describe this as an absorbing Markov chain with 4 states $\{(1,1),(1,0),(0,1),(0,0)\}$ where $(1,1)$ correspons to both controls working, $(1,0),(0,1)$ corresponds to the first or last control working and $(0,0)$ corresponds to the case with both controls failing at the same time.

- (1) Order the states as above and let $(0,0)$ correspond to an absorbing state, construct the transition matrix P for this Markov chain.

Since $(0,0)$ is absorbing and this state can be reached from any other state, the machine will obviously eventually fail. We can now find the probability that that both controls fails withing n steps by considering this Markov chain.

- (1) What is the probability that starting with 2 "tested" controls (state $(1,1)$) the machine fails (is in $(0,0)$) in 3 steps? ($P(X_3 = (0,0)|X_0 = (1,1))$).

Next we want to find expected time until the machine fails. As we ordered the states P should already be in canonical form. We then remember that we could divide our matrix P in blocks:

$$P = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

Where X, Z are aquare matrices. We can then find the average number of steps until absorbtion (and thus the machine failing) as:

$$(I - X)^{-1}\vec{e}$$

Where $\vec{e}$ is the one vector.

- (1) What is the expected number of steps until the machine fails if we start with two tested controls?

Next we consider a slightly different problem. Now the times which the controls needs to be activated occur between regular intervals of the same length. We consider the discrete time points $t_1, t_2, \ldots$ as the points i time between intervals. If the machine fails at one point it is stopped during the whole next interval during which it is repaired and properly tested such that in can be considered to be in in the "tested" state after.

To accomodate this change we change our transition matrix such that we always move to $(1,1)$ after we get to $(0,0)$. If we have the states in the same order as earlier we can do this simply by changing the last row to [1 0 0 0].

Now this new t
ransition matrix is irreducible and primitive and we can use for example Perron-Frobenius to find it's stationary distribution.

- (1) Find the stationary distribution of our new Markov chain, What is the probability of the machine being in the down state $(0,0)$ after a long time?

We assing a reward $r > 0$ whenever the machine is running (all except state $(0,0)$) and a cost $c \geq 0$ to repair a failed 1 control regardless whether the machine is running or not. We then get total reward R of one step as:

$$R(X_n) = \begin{cases} r, & X_n = (1,1) \\ r - c, & X_n \in \{(1,0),(0,1)\} \\ -2c, & X_n = (0,0) \end{cases}$$

We can then find the expected reward per step after a long time as $E(X_n) = \sum_{i=1}^{4} R(\pi_i)$ where π is the stationary distribution. If we only had one control A we would get another Markov chain Y with stationary distribution $\pi_Y = [0.909\ 0.091]^T$ and expected reward $E(Y_n) = 0.909r - 0.091c$

- (1) Find the long term expected reward of one step in the 2 control system. At what values $c, r > 0$ would it be more profitable to use the 2 control system over the 1 control system?