Applied Matrix Analysis, MAA704: Repetition and exercises

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1 Repetition

This section contains a series of exercises in concerning matrices and linear equation systems. Any student that can find the correct answers to these exercises is sufficiently prepared to study MMA704. Some exercises have answers or solutions that can be found in the last section. It is recommended to have a book (or lecture notes) on basic linear algebra available when doing these exercises, some repetition might reduce the amount of calculations significantly.

1.1 Basic matrix notation and operations

notation,addition,subtraction,multiplication of matrices

1.1.1

We have:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$$

a) What is the size of A?

b) What is $a_{2,3}$?

c) What is $b_{2,1}$?

d) Find the diagonal elements of B.

e) What is the trace of A? (the sum of the elements on the diagonal).

1.1.2

We have:

$$A = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & -3 & 4 \\ -2 & 1 & 0 & 1 \\ 0 & -1 & 1 & 3 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Calculate:

a) transpose: $A^\top$

b) addition: $A + B$.

c) subtraction: $A - C$.

d) multiplication AI.

e) multiplication IA.

f) multiplication: AB.

g) multiplication: BA.

h) What is I called.

i) Is matrix multiplication commutative ($AB = BA$ for all matrices A, B)?

1.1.3

We have:

$$A = \begin{bmatrix} 2 & 2 & 7 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Try to do them without calculating the products. Which elements of the following products are non-zero?

a) AA

b) $AA^\top$

c) AB

d) AC

e) BC

1.1.4

We have:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 3 \\ 0 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Which of the following matrix multiplications can we compute and what is the size of the resulting matrix?

a) AA

b) BB

c) AC

d) CA

e) AB

f) BA

g) $AA^\top$

1.1.5

We have:

$$A = \begin{bmatrix} 2 & 2 & 7 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

Which of the above matrices are:

a) Diagonal (all elements except those on the diagonal are zero)?

b) Triangular (all elements above or below the diagonal are zero)?

c) Symmetric ($M = M^\top$)?

1.2 Linear equation systems, determinants and inverses

1.2.1

Given the following linear system:

$$\begin{cases} \text{-2 x+3y=3} \\ \text{x + 2y = 7} \end{cases}$$

a) Write the system using matrices.

b) Solve the system.

1.2.2

a) Show that this linear equation system has one single solution.

$$\begin{cases} -x + 2y - z = 2 \\ -x + y = 3 \\ 2x - y + 2z = 1 \end{cases}$$

b) Show that this linear equation system has infinitely many solutions.

$$\begin{cases} x + y - z = 1 \\ 2x + y + 2z = 2 \\ 3x + 2y - z = 3 \end{cases}$$

c) Show that this linear equation system has no solution.

$$\begin{cases} x - 2y + z = 3 \\ x + 3y - 2z = 2 \\ 2x + y - z = 1 \end{cases}$$

1.2.3

a) How many solutions does this linear equation system have?

$$\begin{cases} & & 2y & + & z & = & 3 \\ 2x & + & 2y & - & z & = & 2 \\ x & & - & & z & = & 1 \end{cases}$$

b) How many solutions does this linear equation system have?

$$\begin{cases} x & + & y & - & 2z & = & 1 \\ -x & & - & & z & = & 2 \\ x & + & y & + & 2z & = & 3 \end{cases}$$

c) How many solutions does this linear equation system have?

$$\begin{cases} -x & + & y & + & 2z & = & 2 \\ -x & + & y & + & z & = & 1 \\ & & & & 2z & = & 3 \end{cases}$$

1.2.4

Calculate these determinants

$$a = \begin{vmatrix} 5 \end{vmatrix}, b = \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix}, c = \begin{vmatrix} 2 & 3 & 1 \\ 2 & -1 & 5 \\ 5 & 0 & -1 \end{vmatrix}$$

1.2.5

Based on what you already know, for which linear equation systems in 1.2.2 and 1.2.3 will the determinant of the corresponding matrix be equal to zero?

1.2.6

Let

$$A = \begin{vmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 2 & 2 & 2 \end{vmatrix}, \ \det(A) = -6$$

$$B = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & -1 \\ 3 & 3 & -1 \end{vmatrix}, \ \det(B) = -2$$

What is

a) $\det(A + B)$

a) $\det(AB)$

1.2.7

Let

$$C = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 3 & 1 & -1 \end{vmatrix}, \ \det(C) = 3$$

and

$$D = \begin{vmatrix} -2 & 0 & 1 \\ 3 & 1 & -1 \\ 1 & 2 & -1 \end{vmatrix}$$

What is $\det(D)$? Try to perform as few calculations as possible.

1.2.8

Note that

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 10$$

What is

$$a = \begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & -1 \end{vmatrix}, \ b = \begin{vmatrix} 3 & 3 & 9 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}, \ c = \begin{vmatrix} 3 & 3 & 9 \\ 2 & -2 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

1.2.9

Let

$$A = \begin{bmatrix} 3 & 3 & 4 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 3 & 4 & 4 \\ 4 & 1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ 2 & 1 & 3 & 3 \end{bmatrix}, C = \begin{bmatrix} 3 & 3 & 4 & 4 \\ 4 & 1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ 3 & 2 & 1 & 3 \end{bmatrix}, \det(C) = -26$$

The following determinants should all be very simple to calculate. Calculate them.

a) $\det(A)$

b) $\det(B)$

c) $\det(C^\top)$

1.2.10

Let

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} -3 & 3 & 3 \\ -9 & 6 & 9 \\ 1 & -1 & 0 \end{bmatrix}$$

then $AB = 3I$.
What is the inverse of A?

1.2.11

Let $M = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ find M^{-1}

1.2.12

If the linear equation systems in 1.2.2 and 1.2.3 were written on the form $Ax = y$, for which systems would A^{-1} exist?

1.3 Eigenvalues and eigenvectors

1.3.1

We have:

$$a = 2, \quad B = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 1 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Calculate:

a) aB

b) CB

c) $CB - aB$

d) Is a an eigenvalue belonging to C?

e) If a is an eigenvalue, is B an eigenvector belonging to a?

1.3.2

Find the eigenvalues and eigenvectors of $M = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

2 Correct answers

1: Basic matrix notation and operations

1.1.1: size $= 3 \times 4$, $a_{2,3} = 3$, $b_{2,1} = 1$, diagonal elements $= 1, 0$, $trace(A) = 1 + 2 + 2 = 5$.
1.1.2: h) Identity matrix, i) Not commutative.
1.1.3: a) diagonal and above, b) all, c) left half, d) top half, e) all.
1.1.4: a) not possible, b) 3×3, c) 3×3, d) 4×4, e) not possible, f) 3×4, g) 3×3.
1.1.5: a) B, b) A, B, c) B, C.

2: Linear equation systems, determinants and inverses

1.2.3: a) infinitely many solutions, b) one solution, c) no solution.
1.2.4: $a = 5$, $b = -2$, $c = 84$.
1.2.5: in 1.2.2: b), c) and in 1.2.3 a) and c).
1.2.6: a) $\det(A + B) = -36$, b) $\det(AB) = 12$.
1.2.7: $\det(D) = 3$.
1.2.8: $a = 20$, $b = 30$, $c = 60$
1.2.9: a) $\det(A) = 9$, b) $\det(B) = 0$, c) $\det(C^\top)$
1.2.10: $A^{-1} = \frac{1}{3}B$
1.2.11: $M^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$
1.2.12: in 1.2.2: a) and in 1.2.3 b).

3: Eigenvalues and eigenvectors

1.3.1: d) a is an eigenvalue and e) B is an eigenvector to a.
1.3.2: With λ denoting eigenvalue and v denoting eigen vector:
$\lambda_1 = 5$, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_1 = 1$, $v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.