Seminar assignment 1

Each seminar assignment consists of two parts, one part consisting of $3-5$ elementary problems for a maximum of 10 points from each assignment. For the second part consisting of problems in applications you are to choose one of four problems to solve. This part can give up to 5 points from each assignment. Each seminar assignment can give a maximum of 15 points, to pass you will need at least 20 points total from both the assignments.

The first part consists of elementary questions to make sure that you have understood the basic material of the course while the second part consists of one larger application example.

Solutions can either be handed in by email to *christopher.engstrom@mdh.se* or *karl.lundengard@mdh.se* or you can hand in handwritten solutions either during the lectures or in the box outside Christopher and Karls room $U3:185$.

1 Part 1

In the first part you are to solve and hand in solutions to the questions. You are allowed to use computer software to check your results, but your calculations as well as your result should be included in the answers for full points.
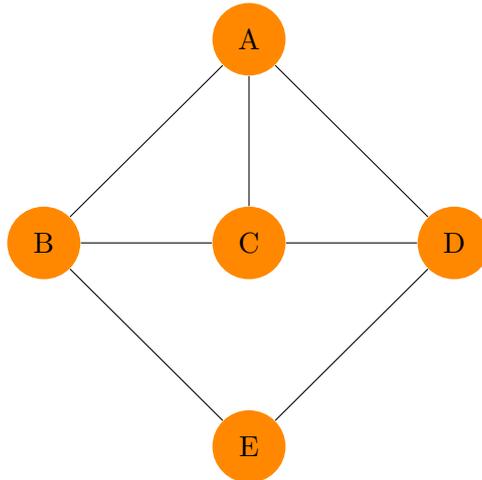
1.1

- (1) Give the definition of eigenvalues and eigenvectors.

- (1) Use the characteristic polynomial $\det(I\lambda - A)$ to find the eigenvalues of:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ 0 & -1 & 0 \end{bmatrix}$$

1.2

Consider the graph:

- (1) What is the Adjacency matrix of the graph.

- (1) What is the Laplacian matrix of the graph.

1.3

We consider the Markov chain on the states (s_1, s_2, s_3, s_4) given by the stochastic matrix P below:

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- (1) Give the definition of a Markov chain.

- (1) Given the stochastic matrix above, what is the graph of the corresponding Markov chain?

- (0.5) If we start in state $X_0 = s_2$ what is the probability to be in state s_2 after 2 steps $(P(X_2 = s_2 | X_0 = s_2))$?

- (0.5) What is the probability to again be in s_2 at step 4 if we were in s_2 at step 2, $(P(X_4 = s_2 | X_2 = s_2))$?

1.4

Consider the matrix A:

$$A = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

- (1) Use the definition of an irreducible matrix to show that A is irreducible.

After calculating the eigenvalues and (left)eigenvectors of A we accidentaly lost which vector belongs to which eigenvalue. We get the eigenvalues as: $\lambda = \{-0.147 \pm 0.50i, \ -0.3724, \ 1\}$ and eigenvectors: as the columns of the matrix below.

$$E = \begin{bmatrix} 0 & 2 & \dfrac{1}{-1+\sqrt{3}i} & \dfrac{1}{-1-\sqrt{3}i} \\ 0 & 1 & \dfrac{1+\sqrt{3}i}{2(-1+\sqrt{3}i)} & \dfrac{1-\sqrt{3}i}{2(-1-\sqrt{3}i)} \\ -1 & 3 & -0.5 & -0.5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Use Perron-Frobenius theorem and answer:

- (1) Which eigenvalue is the Perron-Frobenius eigenvalue?

- (1) Which eigenvector is the Perron vector belonging to the Perron-Frobenius eigenvalue?

2 Part 2

In this second part you are to choose **ONE** example where you attempt to solve the questions presented. If you hand in answers to more than one choice you will get points corresponding to the choice which would give the least total points.

You are allowed to use computer software (but it shouldn't be needed) to solve the questions as long as you make it clear what you are doing and give an example of what could be used to solve it by hand. For example you could write "I solved the linear system $Ax = b$ using Matlabs "fsolve", another method would have been to use Gaussian elimination and solving the found triangular system".

2.1 Voltage and current in electrical networks

Assuming we have a connected graph with edges corresponding to resistors and assign a fixed current v_a to vertice a and $v_b = 0$ to vertice b. Our goal is to use a Markov chain approach to find the voltage v_i of the other vertices as well as the current I_{ij} along the edges of the graph.

We look at the graph above where we put a 1-volt battery across points A and B such that we get a voltage $v_a = 1, \ v_b = 0$. To find the voltage we will look at a random walk on the graph where A, B are absorbing states and the transition probabilities for all other vertices i are decided by the resistors such that.

$$P_{ij} = \frac{C_{ij}}{\sum_{j=1}^{n} C_{ij}}, i \neq A, B$$
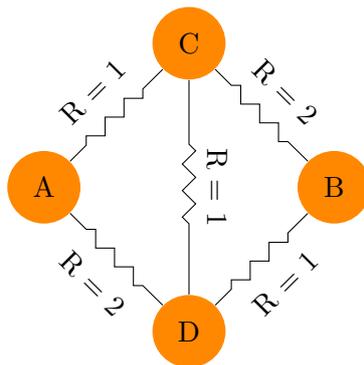
$$C_{ij} = \frac{1}{R_{ij}}$$

- (2) Contruct the new graph and corresponding transition matrix where the transition probabilities are given above. Remember that state A and B are set as absorbing states.

We then look at the random walk on the graph starting in vertice i. The voltage v_i can then be seen as the probability that we end up in a rather than b after absorbtion in one of the states a, b. We can calculate these hitting probabilities h_i by setting up the linear system:

$$h_i = \sum_{j=1}^{n} P_{ij} h_j, \ i \notin \{a, b\}$$

$$h_a = 1$$

$$h_b = 0$$

This can be written in matrix form as a linear system $h = Ah + c$, which can then be solved using any method of choice.

- (1) Set up the linear system and solve it to find the hitting probabilities h.

The voltage is calculated as $v_i = v_a h_i$ which in our case is just the hitting probability $v_i = h_i$.

The current along an edge I_{ij} can be found as the expected number of times to go from i to j when we start in a and only let b be an absorbing state. This however can also be found using Ohm's Law:

$$I_{ij} = \frac{v_i - v_j}{R_{ij}}$$

(Note that $I_{ij} = -I_{ji}$)

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- (1) Calculate the current over the edges and write the graph, noting currents with their value and direction on the edges and voltages on the vertices.

Kirchhoff's Current Law states that for $i \notin \{a, b\}$:

$$\sum_{j=1}^{n} I_{ij} = 0$$

In other words the flow into a vertice is the same as the flow out of a vertice, except for vertices on the border where we fix the voltage.

- (1) Use Ohm's Law and Kirchoff's Current Law to show that $v_i = \sum_{j=1}^{n} P_{ij} v_j, \ i \notin \{a, b\}$ and thus is proportional to the hitting probabilities h_i.

2.2 PageRank

Pagerank is the method originally designed by Google to rank homepages in their search engine in order to show the most relevant pages first. In the ranking algorithm we consider homepages as nodes in a directed graph, and a link between two homepages as a link between the corresponding nodes in the graph. We want pages with links from many other pages and/or links from other important pages to be considered more important. We consider the graph below:
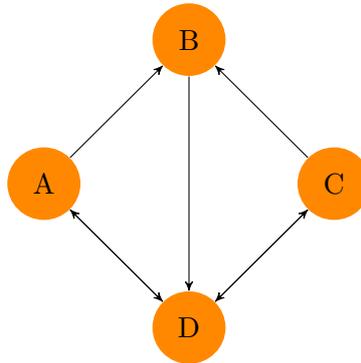
To calculate PageRank we work with a modified adjacency matrix. To start we assume our graph is strongly connected (and our adjacency matrix thus is irreducible). We start by normalizing the adjacency matrix A such that every row sums to 1, we call this A_1. We then choose $0 < c < 1$ in our case $c = 0.85$ and v a vector with elements $v_i = 1/n$, where n is the number of vertices. Using this we create a matrix M as:

$$M = cA_1^T + (1 - c)v1^T$$

- (1) Calculate M for the graph above. (Feel free to keep c rather than substituting $c = 0.85$ in the matrix if you want).

- (1) Now M is a column-stochastic matrix and we can calculate for example it's stationary distribution. Verify that M is a column-stochastic matrix and show that M is a (column) stochastic irreducible matrix for any irreduvible row-stochastic matrix A, given our choice of c, v.

PageRank is defined as the (right) eigenvector with eigenvalue 1 for M:

$$\pi = M\pi$$

Normalized such that $\sum_{i=1}^{n} \pi_i = 1$.

If we consider a random walk on the graph where we in every step move to one of the nodes linked to from our current node with probability c or start in a new random node with probability $1 - c$. Then PageRank can be seen as the probability distribution to be in the nodes after a long time.

- (1) Find PageRank for the graph above described by M. (You can probably save some time by making the calculations using for example Maple or Matlab).

In reality we don't require A to be irreducible, instead we forbid any node to link to itself, and make a slight adjustment to the matrix for nodes which have no link to any other node (represented by a row of all zeros in A). We let these so called dangling nodes link to all nodes(itself included) with equal probability. We consider adjacency matrix A below:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- (1) Make the necessary adjustments of A and find our new matrix M.

To calculate PageRank we usually use a method called the Power method which iterates $x_{n+1} = Mx_n$ until convergence. This can be seen as iterating the Markov chain described by M until we reach it's stationary distribution.

- (1) Show that given our choice of c, v then M is a positive matrix and use Perron-Frobenius to show that the Markov chain described by M always converge to it's stationary distribution (and the Power method thus is guaranteed to converge). (Note that M is column stochastic rather than row stochastic as we are used to when working with Markov chains).

2.3 Credit rating of bonds

We consider how the credit rating of bonds change from one rating to another or defaulting. The bonds are rated once every year at the same time. From historical data we have found that the credit rating of a bond can be modelled using a markov chain with transition matrix P. We consider the 3 ratings A, B, C and default D. We note that this is a simplified model in order to allow calculations by hand, in reality there should be a possibility to move between $A \leftrightarrow D$ as well as there possibly being more ratings such as A, AA, AAA.

$$P = \begin{bmatrix} 0.94 & 0.05 & 0.01 & 0.00 \\ 0.05 & 0.85 & 0.05 & 0.05 \\ 0 & 0.15 & 0.6 & 0.25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- (1) Write the graph for the Markov chain described by P.

Often we are interested in not only how likely a bond is to say default during the next year, but also over many years. While it's true that a bond rated B seems to be 5 times more likely to default than a bond rated A in one year. The same might not be true if we consider the bond over 3 years.

- (1) What is the probability that a B rated bond will be in default after 3 years? $(P(X_3 = D | X_0 = B))$.

While it's obvious that all bonds in our model will eventually default, we are interested in how long we can expect it to take. We remember we could find the hitting time k_i as the minimal solution to:

$$\begin{cases} k_i^A = 0, & i \in A \\ k_i = 1 + \sum_{j=1}^{n} p_{ij} k_j^A & i \notin A \end{cases}$$

- (1) Find the expected time until default for a bond starting in B, (k_B). You can either set up the linear system $k = Ak + c$ to solve it on the computer, or solve the system by hand.

The model above have one problem when we want to look at all the bonds, every bond is bound to eventually default and since we never introduce new bonds, eventually all bonds will have defaulted. But we want to say something about the long term as well. We consider the model where we always have the same number of bonds. We do this by creating a new bond every time one defaults. At the same time we remove the defaulting bond since it is for us no longer interesting. We let the new bond start with any rating with equal probability. This gives the new transition matrix below:

$$P' = \begin{bmatrix} 0.94 & 0.05 & 0.01 & 0.00 \\ 0.05 & 0.85 & 0.05 & 0.05 \\ 0 & 0.15 & 0.6 & 0.25 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

Now we can talk about the long term behaviour of the chain as well as answer questions such as, how large percentage of total bonds can we expect to be A rated.

- (1) Show that P' is primitive.

- (1) What is the stationary distribution of this new Markov chain described by P'? Assuming a constant 100 bonds, how many can we expect to default every year after a long time?

2.4 Fail-Safe system

Consider a machine with two independent controls A, B designed to immediately stop the machine if something bad is about to happen. Whenever the machine is stopped by A or B the problem can quickly be fixed and the machine restarted almost immediately. However if both controls fails to stop the machine at the same time something bad have happened to either the machine or the operator. We consider the discrete time points $t_1, t_2, \ldots$ where the machine needs to be stopped.

If one control fails to activate it is immediately replaced with a new one before the next time point. A control that worked during the last time of activation t_n have a 90% chance to work at the next time it needs to activate t_{n+1}. However immediately after a control fails, the new untested replacement have only a 75% to activate at the next time step, if it works correctly in that step it's considered "tested" and works with 90% reliability after that.

We describe this as an absorbing Markov chain with 4 states $\{(1,1), (1,0), (0,1), (0,0)\}$ where $(1,1)$ correspons to both controls working, $(1,0), (0,1)$ corresponds to the first or last control working and $(0,0)$ corresponds to the case with both controls failing at the same time.

- (1) Order the states as above and let $(0,0)$ correspond to an absorbing state, construct the transition matrix P for this Markov chain.

Since $(0,0)$ is absorbing and this state can be reached from any other state, the machine will obviously eventually fail. We can now find the probability that that both controls fails withing n steps by considering this Markov chain.

- (1) What is the probability that starting with 2 "tested" controls (state $(1,1)$) the machine fails (is in $(0,0)$) in 3 steps? ($P(X_3 = (0,0)|X_0 = (1,1))$).

Next we want to find expected time until the machine fails. As we ordered the states P should already be in canonical form. We then remember that we could divide our matrix P in blocks:

$$P = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

Where X, Z are aquare matrices. We can then find the average number of steps until absorbtion (and thus the machine failing) as:

$$(I - X)^{-1} \vec{e}$$

Where $\vec{e}$ is the one vector.

- (1) What is the expected number of steps until the machine fails if we start with two tested controls?

Next we consider a slightly different problem. Now the times which the controls needs to be activated occur between regular intervals of the same length. We consider the discrete time points $t_1, t_2, \ldots$ as the points i time between intervals. If the machine fails at one point it is stopped during the whole next interval during which it is repaired and properly tested such that in can be considered to be in in the "tested" state after.

To accomodate this change we change our transition matrix such that we always move to $(1, 1)$ after we get to $(0, 0)$. If we have the states in the same order as earlier we can do this simply by changing the last row to $[1\ 0\ 0\ 0]$.

Now this new t
ransition matrix is irreducible and primitive and we can use for example Perron-Frobenius to find it's stationary distribution.

- (1) Find the stationary distribution of our new Markov chain, What is the probability of the machine being in the down state $(0, 0)$ after a long time?

We assing a reward $r > 0$ whenever the machine is running (all except state $(0, 0)$) and a cost $c \geq 0$ to repair a failed 1 control regardless whether the machine is running or not. We then get total reward R of one step as:

$$R(X_n) = \begin{cases} r, & X_n = (1, 1) \\ r - c, & X_n \in \{(1, 0), (0, 1)\} \\ -2c, & X_n = (0, 0) \end{cases}$$

We can then find the expected reward per step after a long time as $E(X_n) = \sum_{i=1}^{4} R(\pi_i)$ where π is the stationary distribution. If we only had one control A we would get another Markov chain Y with stationary distribution $\pi_Y = [0.909\ 0.091]^T$ and expected reward $E(Y_n) = 0.909r - 0.091c$

- (1) Find the long term expected reward of one step in the 2 control system. At what values $c, r > 0$ would it be more profitable to use the 2 control system over the 1 control system?