Exam 1, MMA302, Solutions

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1. Determine the following limit (if it exists).

$$\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta}.$$

Solution.

$$\begin{aligned}\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta} &= \lim_{\theta\to 0}\frac{\cos\theta\sin\theta}{\theta\cos\theta}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\\ &= 1.\end{aligned}$$

Answer: 1. □

2. Evaluate the derivative of the implicit function $(x^2+y^2)^2 = 4x^2y$ at the point $(1,1)$.

Solution.

$$\begin{aligned}(x^2+y^2)^2 &= 4x^2y,\\ \frac{d}{dx}(x^2+y^2)^2 &= \frac{d}{dx}(4x^2y),\\ 2(x^2+y^2)(2x+2y\frac{dy}{dx}) &= 4(2xy+x^2\frac{dy}{dx}),\\ (x^2+y^2)(x+y\frac{dy}{dx}) &= 2xy+x^2\frac{dy}{dx},\\ x^3+x^2y\frac{dy}{dx}+xy^2+y^3\frac{dy}{dx} &= 2xy+x^2\frac{dy}{dx},\\ \frac{dy}{dx}(x^2y+y^3-x^2) &= 2xy-x^3-xy^2,\\ \frac{dy}{dx} &= \frac{2xy-x^3-xy^2}{x^2y+y^3-x^2},\\ \frac{dy}{dx}(1,1) &= \frac{2\cdot 1\cdot 1-1^3-1\cdot 1^2}{1^2\cdot 1+1^3-1^2}\\ &= 0.\end{aligned}$$

Answer: 0. □

3. Evaluate the definite integral

$$\int_0^2 e^{-x} \cos x \, dx.$$

Solution. Put

$$u(x) = \cos x, \qquad dv(x) = e^{-x}\, dx.$$

Then,

$$du(x) = -\sin x \, dx, \qquad v(x) = -e^{-x}\, dx,$$

and

$$\begin{aligned} \int_0^2 e^{-x} \cos x \, dx &= -e^{-x} \cos x|_0^2 - \int_0^2 (-e^{-x})(-\sin x)\, dx \\ &= -e^{-2} \cos 2 + 1 - \int_0^2 e^{-x} \sin x \, dx. \end{aligned}$$

Put

$$u(x) = \sin x, \qquad dv(x) = e^{-x}\, dx.$$

Then,

$$du(x) = \cos x \, dx, \qquad v(x) = -e^{-x}\, dx,$$

and

$$\begin{aligned} \int_0^2 e^{-x} \cos x \, dx &= -e^{-2} \cos 2 + 1 + e^{-x} \sin x|_0^2 - \int_0^2 e^{-x} \cos x \, dx \\ &= -e^{-2} \cos 2 + 1 + e^{-2} \sin 2 - \int_0^2 e^{-x} \cos x \, dx. \end{aligned}$$

It follows that

$$2 \int_0^2 e^{-x} \cos x \, dx = e^{-2} \sin 2 - e^{-2} \cos 2 + 1$$

and

$$\int_0^2 e^{-x} \cos x \, dx = \frac{1}{2}(e^{-2} \sin 2 - e^{-2} \cos 2 + 1).$$

Answer: $\frac{1}{2}(e^{-2} \sin 2 - e^{-2} \cos 2 + 1)$. □

4. Find the particular solution of the differential equation that satisfies the boundary condition.

$$x^3 y' + 2y = e^{1/x^2}, \qquad y(1) = e.$$

Solution. To write this equation in standard form, divide both hand sides by x^3:

$$y' + 2x^{-3} y = x^{-3} e^{1/x^2}.$$

We have $P(x) = 2x^{-3}$ and $Q(x) = x^{-3}e^{1/x^2}$. Then

$$\begin{aligned}\int P(x)\,dx &= \int 2x^{-3}\,dx \\ &= -x^{-2}.\end{aligned}$$

The integrating factor is

$$\begin{aligned}u(x) &= e^{\int P(x)\,dx} \\ &= e^{-1/x^2}.\end{aligned}$$

Multiply both hand sides of the standard form by the integrating factor. We have

$$\begin{aligned}y'e^{-1/x^2} + 2x^{-3}e^{-1/x^2}y &= x^{-3}, \\ \frac{d}{dx}(ye^{-1/x^2}) &= x^{-3}, \\ ye^{-1/x^2} &= \int x^{-3}\,dx, \\ ye^{-1/x^2} &= -\frac{1}{2x^2} + C, \\ y &= e^{1/x^2}(C - \frac{1}{2x^2}).\end{aligned}$$

To find C, substitute the values $x = 1$ and $y = e$. We obtain

$$\begin{aligned}e &= e(C - \frac{1}{2}), \\ 1 &= C - \frac{1}{2}, \\ C &= 3/2.\end{aligned}$$

The particular solution is

$$\begin{aligned}y &= e^{1/x^2}(\frac{3}{2} - \frac{1}{2x^2}) \\ &= \frac{e^{1/x^2}(3x^2 - 1)}{2x^2}.\end{aligned}$$

Answer: $y = \dfrac{e^{1/x^2}(3x^2 - 1)}{2x^2}$. □

5. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n.$$

Solution. We have

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left|\frac{2n}{n+1}\right|^{n/n} \\
&= \lim_{n\to\infty} \frac{2n}{n+1} \\
&= \lim_{n\to\infty} \frac{2n/n}{(n+1)/n} \\
&= \lim_{n\to\infty} \frac{2}{1+1/n} \\
&= 2 > 1.
\end{aligned}$$

Answer: The series diverges. □