# Exam 2, MMA302, Solutions

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1. Determine the following limit (if it exists).

$$\lim_{x\to 0}\frac{2\tan^2 x}{x}.$$

*Solution.*

$$\begin{aligned}\lim_{x\to 0}\frac{2\tan^2 x}{x} &= \lim_{x\to 0}\frac{2\sin^2 x}{x\cos^2 x}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\frac{2\sin x}{\cos^2 x}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{2\sin x}{\cos^2 x}\\ &= 1\cdot\frac{2\cdot 0}{1}\\ &= 0.\end{aligned}$$

*Answer*: 0. □

2. Evaluate the derivative of the implicit function $(4-x)y^2 = x^3$ at the point $(2,2)$.

*Solution.*

$$
\begin{aligned}
(4-x)y^2 &= x^3 \\
\frac{d}{dx}[(4-x)y^2] &= \frac{d}{dx}[x^3] \\
\frac{d}{dx}[4-x]y^2 + (4-x)\frac{d}{dx}[y^2] &= 3x^2 \\
-y^2 + (4-x)2y\frac{dy}{dx} &= 3x^2 \\
\frac{dy}{dx}2y(4-x) &= 3x^2 + y^2 \\
\frac{dy}{dx} &= \frac{3x^2+y^2}{2y(4-x)} \\
\frac{dy}{dx}(2,2) &= \frac{3\cdot 2^2 + 2^2}{2\cdot 2(4-2)} \\
\frac{dy}{dx}(2,2) &= 2.
\end{aligned}
$$

*Answer*: 2. □

3. Evaluate the definite integral

$$\int_0^1 \ln(1+x^2)\,dx.$$

*Solution.* Put

$$u(x) = \ln(1+x^2), \qquad dv(x) = dx.$$

Then,

$$du(x) = \frac{2x\,dx}{1+x^2}, \qquad v(x) = x,$$

and

$$
\begin{aligned}
\int_0^1 \ln(1+x^2)\,dx &= x\ln(1+x^2)\big|_0^1 - \int_0^1 \frac{2x^2}{1+x^2}\,dx \\
&= 1\cdot\ln(1+1^2) - 0\cdot\ln(1+0^2) - \int_0^1 \frac{2x^2+2-2}{1+x^2}\,dx \\
&= \ln 2 - \int_0^1 \frac{2x^2+2}{1+x^2}\,dx + \int_0^1 \frac{2}{1+x^2}\,dx \\
&= \ln 2 - \int_0^1 2\,dx + 2\int_0^1 \frac{dx}{1+x^2} \\
&= \ln 2 - 2 + 2\arctan x\big|_0^1 \\
&= \ln 2 - 2 + 2\arctan 1 - 2\arctan 0 \\
&= \ln 2 - 2 + \pi/2.
\end{aligned}
$$

*Answer*: $\ln 2 + \pi/2 - 2$. □

4. Find the particular solution of the differential equation that satisfies the boundary condition.

$$y' + y\sec x = \sec x, \qquad y(0) = 4.$$

*Solution.* We have $P(x) = Q(x) = \sec x$. Then,

$$\begin{aligned}\int P(x)\,dx &= \int \sec x\,dx \\ &= \ln|\sec x + \tan x|.\end{aligned}$$

The integrating factor is

$$\begin{aligned}u(x) &= e^{\int P(x)\,dx} \\ &= e^{\ln|\sec x + \tan x|} \\ &= |\sec x + \tan x| \\ &= \sec x + \tan x,\end{aligned}$$

because

$$\begin{aligned}\sec x + \tan x &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \\ &= \frac{1 + \sin x}{\cos x} > 0\end{aligned}$$

in a neighbourhood of 0. Multiply both hand sides of the equation by $u(x)$. We obtain

$$\begin{aligned}y'(\sec x + \tan x) + y\sec x(\sec x + \tan x) &= \sec x(\sec x + \tan x), \\ \frac{d}{dx}[y(\sec x + \tan x)] &= \sec^2 x + \sec x \tan x, \\ y(\sec x + \tan x) &= \int \sec^2 x\,dx + \int \sec x \tan x\,dx + C, \\ y(\sec x + \tan x) &= \tan x + \sec x + C, \\ y &= \frac{\tan x + \sec x + C}{\sec x + \tan x} \\ &= 1 + \frac{C}{\sec x + \tan x}.\end{aligned}$$

To find $C$, use the following.

$$\begin{aligned}y(0) &= 4, \\ 1 + \frac{C}{\sec 0 + \tan 0} &= 4, \\ 1 + \frac{C}{1 + 0} &= 4, \\ C &= 3.\end{aligned}$$

*Answer*: $y = 1 + \frac{3}{\sec x + \tan x}$. □

5. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} \left( \frac{-3n}{2n+1} \right)^{3n}.$$

*Solution.* We have

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left| \frac{-3n}{2n+1} \right|^{3n/n} \\
&= \lim_{n\to\infty} \left( \frac{3n}{2n+1} \right)^3 \\
&= \lim_{n\to\infty} \left( \frac{3n/n}{2n/n+1/n} \right)^3 \\
&= \lim_{n\to\infty} \left( \frac{3}{2+1/n} \right)^3 \\
&= \left( \frac{3}{2+0} \right)^3 > 1.
\end{aligned}$$

*Answer*: the series diverges. □