Exam 3, MMA302, Solutions

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1. Determine the following limit (if it exists).

$$\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}.$$

Solution.

$$\begin{aligned}
\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x} &= \lim_{x\to\pi/4}\frac{1-\sin x/\cos x}{\sin x-\cos x}\\
&= \lim_{x\to\pi/4}\frac{(1-\sin x/\cos x)\cos x}{(\sin x-\cos x)\cos x}\\
&= \lim_{x\to\pi/4}\frac{\cos x-\sin x}{(\sin x-\cos x)\cos x}\\
&= -\lim_{x\to\pi/4}\frac{1}{\cos x}\\
&= -\frac{1}{1/\sqrt{2}}\\
&= -\sqrt{2}.
\end{aligned}$$

Answer: $-\sqrt{2}$. □

2. Evaluate the derivative of the implicit function $(x^2+4)y=8$ at the point $(2,1)$.

Solution.

$$\begin{aligned}
(x^2+4)y &= 8\\
\frac{d}{dx}[(x^2+4)y] &= \frac{d}{dx}[8]\\
2xy+(x^2+4)\frac{dy}{dx} &= 0\\
\frac{dy}{dx} &= -\frac{2xy}{x^2+4}\\
&= -\frac{2\cdot 2\cdot 1}{2^2+4}\\
&= -1/2.
\end{aligned}$$

Answer: $-1/2$. $\square$

3. Evaluate the definite integral

$$\int_0^{\pi/4} x\sec^2 x\,dx.$$

Solution. Put $u = x$, $dv = \sec^2 x\,dx$. Then $du = dx$, $v = \tan x$ and

$$\begin{aligned}\int_0^{\pi/4} x\sec^2 x\,dx &= x\tan x|_0^{\pi/4} - \int_0^{\pi/4} \tan x\,dx \\ &= \pi/4 + \int_0^{\pi/4} \frac{-\sin x}{\cos x}\,dx \\ &= \pi/4 + (\ln|\cos x|)|_0^{\pi/4} \\ &= \pi/4 - (\ln 2)/2.\end{aligned}$$

Answer: $\pi/4 - (\ln 2)/2$. $\square$

4. Find the particular solution of the differential equation that satisfies the boundary condition.

$$y' + (2x-1)y = 0, \qquad y(1) = 2.$$

Solution. We have $P(x) = 2x-1$. Then

$$\int P(x)\,dx = \int (2x-1)\,dx = x^2 - x.$$

The integrating factor is

$$u(x) = e^{\int P(x)\,dx} = e^{x^2-x}.$$

Multiply both hand sides of the equation by $u(x)$. We obtain

$$\begin{aligned} y'e^{x^2-x} + e^{x^2-x}(2x-1)y &= 0 \\ \frac{d}{dx}(e^{x^2-x}y) &= 0 \\ e^{x^2-x}y &= C \\ y &= Ce^{x-x^2}.\end{aligned}$$

To find C, use the following.

$$\begin{aligned} y(1) &= 2 \\ Ce^{1-1^2} &= 2 \\ C &= 2 \\ y &= 2e^{x-x^2}.\end{aligned}$$

Answer: $y = 2e^{x-x^2}$. $\square$

5. Use the Root Test to determine the convergence or divergence of the series.

$$\sum_{n=1}^{\infty} e^{-n}.$$

Solution. We have

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} (e^{-n})^{1/n} \\ &= e^{-1} < 1.\end{aligned}$$

Answer: the series converges. □