# Exam 2, MMA302, Solutions

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1. Evaluate the limit or explain why it does not exist.
$$\lim_{x\to -1}\frac{x^3+1}{x+1}.$$

*Solution.*
$$\begin{aligned}\lim_{x\to -1}\frac{x^3+1}{x+1} &= \lim_{x\to -1}\frac{(x+1)(x^2-x+1)}{x+1}\\ &= \lim_{x\to -1} x^2-x+1\\ &= (-1)^2-(-1)+1\\ &= 3.\end{aligned}$$

*Answer*: 3. □

2. Let $y=\sec x$.

a) Find $y'$ (1 p.)

b) Find $y''$ (1 p.)

c) Find $y'''$ (2 p.)

*Solution.* The first derivative
$$y'=\sec x\tan x.$$

The second derivative
$$\begin{aligned}y'' &= (\sec x\tan x)'\\ &= (\sec x)'\tan x+\sec x(\tan x)'\\ &= \sec x\tan x\tan x+\sec x\sec^2 x\\ &= \sec x\tan^2 x+\sec^3 x.\end{aligned}$$

The third derivative
$$\begin{aligned}y''' &= (\sec x\tan^2 x+\sec^3 x)'\\ &= (\sec x)'\tan^2 x+\sec x\cdot 2\tan x(\tan x)'+3\sec^2 x(\sec x)'\\ &= \sec x\tan x\tan^2 x+2\sec x\tan x\sec^2 x+3\sec^2 x\sec x\tan x\\ &= \sec x\tan^3 x+5\sec^3 x\tan x.\end{aligned}$$

*Answer*: $y' = \sec x \tan x$, $y'' = \sec x \tan^2 x + \sec^3 x$, $y''' = \sec x \tan^3 x + 5\sec^3 x \tan x$. ☐

3. Evaluate the integral.

$$\int x^2 \tan^{-1} x \, \mathrm{d}x.$$

*Solution.* Put $u = \tan^{-1} x$. Then $\mathrm{d}v = x^2\,\mathrm{d}x$, $\mathrm{d}u = \dfrac{\mathrm{d}x}{x^2+1}$, and $v = \dfrac{x^3}{3}$. Then

$$\begin{aligned}
\int x^2 \tan^{-1} x \, \mathrm{d}x &= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{x^2+1}\,\mathrm{d}x \\
&= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3+x-x}{x^2+1}\,\mathrm{d}x \\
&= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \left(\frac{x^3+x}{x^2+1} - \frac{x}{x^2+1}\right)\mathrm{d}x \\
&= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \left(x - \frac{1}{2}\frac{2x}{x^2+1}\right)\mathrm{d}x \\
&= \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{\ln(x^2+1)}{6} + C.
\end{aligned}$$

*Answer*: $\dfrac{x^3}{3}\tan^{-1} x - \dfrac{x^2}{6} + \dfrac{\ln(x^2+1)}{6} + C$. ☐

4. Solve the separable differential equation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2.$$

*Solution.*

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 + y^2, \\
\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{1+y^2} &= (1+y^2)\frac{\mathrm{d}x}{1+y^2}, \\
\frac{\mathrm{d}y}{1+y^2} &= \mathrm{d}x, \\
\int \frac{\mathrm{d}y}{1+y^2} &= \int \mathrm{d}x, \\
\tan^{-1} y &= x + C, \\
y &= \tan(x+C).
\end{aligned}$$

*Answer*: $y = \tan(x+C)$. ☐

5. Consider the following power series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n.$$

a) Determine the centre of convergence (1 p.)

b) Determine the radius of convergence (1 p.)

c) Determine the interval of convergence (2 p.)

*Solution.* a) Compare the series with the standard form

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

We see that $c = 0$.

b) Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)^4 2^{2(n+1)}} \cdot \frac{n^4 2^{2n}}{(-1)^n x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^n \cdot x}{(n+1)^4 2^{2n} \cdot 2^2} \cdot \frac{n^4 2^{2n}}{x^n} \right| \\
&= \left| \frac{x}{4} \right| \lim_{n\to\infty} \left( \frac{n}{n+1} \right)^4 \\
&= \left| \frac{x}{4} \right|.
\end{aligned}$$

The series converges absolutely if $|x/4| < 1$, or $-1 < x/4 < 1$, or $-4 < x < 4$. The series diverges when $|x/4| > 1$, or $x \in (-\infty, -4) \cup (4, \infty)$. The radius of convergence is $R = 0 - (-4) = 4$.

c) At $x = -4$, the series is

$$\sum_{n=0}^{\infty} \frac{1}{n^4},$$

which converges by the $p$-series test. At $x = 4$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4},$$

which converges by the Alternating Series Test. Hence, the interval of convergence is $[-4, 4]$.

*Answer*: $c = 0$, $R = 4$, $[-4, 4]$. □