Exam 3, MMA302, Solutions

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1. Evaluate the limit or explain why it does not exist.

$$\lim_{x\to 8}\frac{x^{2/3}-4}{x^{1/3}-2}.$$

Solution.

$$\begin{aligned}\lim_{x\to 8}\frac{x^{2/3}-4}{x^{1/3}-2}&=\lim_{x\to 8}\frac{(x^{1/3}+2)(x^{1/3}-2)}{x^{1/3}-2}\\&=\lim_{x\to 8}(x^{1/3}+2)\\&=8^{1/3}+2\\&=4.\end{aligned}$$

Answer: 4. □

2. Let $y=\dfrac{\sin x}{x}$.

a) Find y' (1 p.)

b) Find y'' (1 p.)

c) Find y''' (2 p.)

Solution. The first derivative

$$\begin{aligned}y'&=\frac{(\sin x)'x-\sin x(x)'}{x^2}\\&=\frac{\cos x\cdot x-\sin x}{x^2}\\&=\frac{\cos x}{x}-\frac{\sin x}{x^2}.\end{aligned}$$

The second derivative

$$
\begin{aligned}
y'' &= \frac{(\cos x)'x - \cos x(x)'}{x^2} - \frac{(\sin x)'x^2 - \sin x(x^2)'}{x^4} \\
&= \frac{-\sin x \cdot x - \cos x}{x^2} - \frac{\cos x \cdot x^2 - \sin x \cdot 2x}{x^4} \\
&= \sin x\left(-\frac{x}{x^2} + \frac{2x}{x^4}\right) + \cos x\left(-\frac{1}{x^2} - \frac{x^2}{x^4}\right) \\
&= \frac{(2-x^2)\sin x}{x^3} - \frac{2\cos x}{x^2}.
\end{aligned}
$$

The third derivative

$$
\begin{aligned}
y''' &= \frac{[(2-x^2)\sin x]'x^3 - [(2-x^2)\sin x](x^3)'}{x^6} - \frac{(2\cos x)'x^2 - 2\cos x(x^2)'}{x^4} \\
&= \frac{[-2x\sin x + (2-x^2)\cos x]x^3 - [(2-x^2)\sin x]3x^2}{x^6} - \frac{-2\sin x \cdot x^2 - 2\cos x \cdot 2x}{x^4} \\
&= \frac{-2x^4\sin x + 2x^3\cos x - x^5\cos x - 6x^2\sin x + 3x^4\sin x}{x^6} - \frac{-2\sin x \cdot x^2 - 2\cos x \cdot 2x}{x^4} \\
&= \sin x\left(-\frac{2x^4}{x^6} - \frac{6x^2}{x^6} + \frac{3x^4}{x^6} + \frac{2x^2}{x^4}\right) + \cos x\left(\frac{2x^3}{x^6} - \frac{x^5}{x^6} + \frac{4x}{x^4}\right) \\
&= \frac{3(x^2-2)\sin x}{x^4} + \frac{(6-x^2)\cos x}{x^3}.
\end{aligned}
$$

Answer: $y' = \frac{\cos x}{x} - \frac{\sin x}{x^2}$, $y'' = \frac{(2-x^2)\sin x}{x^3} - \frac{2\cos x}{x^2}$, $y''' = \frac{3(x^2-2)\sin x}{x^4} + \frac{(6-x^2)\cos x}{x^3}$. □

3. Evaluate the integral

$$\int_0^1 x^2 \sin(\pi x)\,\mathrm{d}x.$$

Solution. Put $u = x^2$. Then $\mathrm{d}v = \sin(\pi x)\,\mathrm{d}x$, $\mathrm{d}u = 2x\,\mathrm{d}x$, and $v = -\frac{1}{\pi}\cos(\pi x)$. Then

$$
\begin{aligned}
\int_0^1 x^2 \sin(\pi x)\,\mathrm{d}x &= -\frac{1}{\pi}x^2\cos(\pi x)\Big|_0^1 - \int_0^1 \left(-\frac{1}{\pi}\right)\cos(\pi x)2x\,\mathrm{d}x \\
&= \frac{1}{\pi} + \frac{2}{\pi}\int_0^1 x\cos(\pi x)\,\mathrm{d}x.
\end{aligned}
$$

Put $u = x$. Then $\mathrm{d}v = \cos(\pi x)\,\mathrm{d}x$, $\mathrm{d}u = \mathrm{d}x$, and $v = \dfrac{1}{\pi}\sin(\pi x)$. Then

$$\begin{aligned}
\int_0^1 x\cos(\pi x)\,\mathrm{d}x &= \frac{1}{\pi}x\sin(\pi x)\bigg|_0^1 - \int_0^1 \left(\frac{1}{\pi}\right)\sin(\pi x)\,\mathrm{d}x \\
&= -\frac{1}{\pi}\int_0^1 \sin(\pi x)\,\mathrm{d}x \\
&= \frac{1}{\pi^2}\cos(\pi x)|_0^1 \\
&= -\frac{2}{\pi^2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_0^1 x^2\sin(\pi x)\,\mathrm{d}x &= \frac{1}{\pi} - \frac{2}{\pi}\frac{2}{\pi^2} \\
&= \frac{\pi^2 - 4}{\pi^3}.
\end{aligned}$$

Answer: $\dfrac{\pi^2 - 4}{\pi^3}$. □

4. Solve the separable differential equation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2(1-y).$$

Solution.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2(1-y), \\
\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{y^2(1-y)} &= y^2(1-y)\frac{\mathrm{d}x}{y^2(1-y)}, \\
\frac{\mathrm{d}y}{y^2(1-y)} &= \mathrm{d}x, \\
\int \frac{\mathrm{d}y}{y^2(1-y)} &= \int \mathrm{d}x.
\end{aligned}$$

Expand the left-hand side in partial fractions.

$$\begin{aligned}
\frac{1}{y^2(1-y)} &= \frac{A}{y} + \frac{B}{y^2} + \frac{C}{1-y}, \\
\frac{y^2(1-y)}{y^2(1-y)} &= \frac{Ay^2(1-y)}{y} + \frac{By^2(1-y)}{y^2} + \frac{Cy^2(1-y)}{1-y}, \\
1 &= Ay(1-y) + B(1-y) + Cy^2.
\end{aligned}$$

Put $y = 0$. Then $B = 1$. Put $y = 1$. Then $C = 1$. Put $y = -1$. Then $1 = A(-1)[1 - (-1)] + 1 \cdot [1 - (-1)] + 1(-1)^2$, or $A = 1$. Hence

$$\begin{aligned}\int \frac{\mathrm{d}y}{y^2(1-y)} &= \int \left(\frac{1}{y} + \frac{1}{y^2} + \frac{1}{1-y} \right) \mathrm{d}y \\ &= \ln|y| - \frac{1}{y} - \ln|1-y| \\ &= \ln\left|\frac{y}{1-y}\right| - \frac{1}{y}.\end{aligned}$$

Therefore,

$$\ln\left|\frac{y}{1-y}\right| - \frac{1}{y} = x + K.$$

Answer: $\ln\left|\frac{y}{1-y}\right| - \frac{1}{y} = x + K.$ □

5. Consider the following power series.

$$\sum_{n=0}^{\infty} \frac{e^n}{n^3}(4-x)^n.$$

a) Determine the centre of convergence (1 p.)

b) Determine the radius of convergence (1 p.)

c) Determine the interval of convergence (2 p.)

Solution. a) Compare the series

$$\sum_{n=0}^{\infty} \frac{e^n}{n^3}(4-x)^n = \sum_{n=0}^{\infty} \frac{(-e)^n}{n^3}(x-4)^n$$

with the standard form

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

We see that $c = 4$.

b) Use the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-e)^{n+1}(x-4)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(-e)^n(x-4)^n}\right| \\ &= e|x-4| \lim_{n\to\infty} \frac{n^3}{(n+1)^3} \\ &= e|x-4|.\end{aligned}$$

The series converges absolutely if $e|x-4| < 1$, or $|x-4| < e^{-1}$, or $4-e^{-1} < x < 4+e^{-1}$. The series diverges when $e|x-4| > 1$, or $x \in (-\infty, 4-e^{-1}) \cup (4+e^{-1}, \infty)$. The radius of convergence is $R = 4-(4-e^{-1}) = e^{-1}$.

c) At $x = 4-e^{-1}$, the series is

$$\sum_{n=0}^{\infty} \frac{1}{n^3},$$

which converges by the p-series test. At $x = 4+e^{-1}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3},$$

which converges by the Alternating Series Test. Hence, the interval of convergence is $[4-e^{-1}, 4+e^{-1}]$.

Answer: $c = 4$, $R = e^{-1}$, $[4-e^{-1}, 4+e^{-1}]$. □